SCALING VIOLATION IN A FIELD THEORY OF CLOSED STRINGS IN ONE PHYSICAL DIMENSION

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Received 28 December 1989

A one dimensional field theory of closed strings is solved exactly in a special double scaling
limit, in which the string coupling $1/N$ goes to zero, the cosmological constant $\Lambda$ approaches a
critical value $\Lambda_c$ (corresponding to the limit of an infinitely large world sheet), and some
nontrivial scaling parameter $\xi(\Lambda, N)$ is fixed. The structure of singularities of the string
susceptibility $\chi(\Lambda, N)$ is analyzed, order by order in the topological $(1/N)$ expansion, as well as
nonperturbatively, for arbitrary $\xi$. It is shown that the "naive" scaling, which confirmed the work
of Polyakov, Knizhnik and Zamolodchikov, of David, and of Distler and Kawai when the central
charge is smaller than one, is violated by logarithmic corrections at every order of the topological
expansion. Nonperturbative effects in $1/N$ arise through vacuum instabilities of this string field
theory for any finite $\xi$.

1. Introduction

A coordinate free formulation of 2D quantum gravity in the presence of matter
fields with central charges $c < 1$, has led to recent advances in the nonperturbative
investigation of string field theories of closed [1–5] and open [6] strings (and a
generalization including both of them [7]). This formulation, which may be called
quantum Regge calculus, relies on the statistical mechanical models of dynamical
triangulations [8–10] classified by their topologies, representing the quantum
fluctuations of the 2D internal metric.

The introduction of matter fields can be achieved in a straightforward way by
means of spin variables, located at the sites of a dynamical lattice. In this manner,
one can introduce a discretized version of Polyakov's bosonic string [10–12].

Many of these models can be solved exactly [1–9, 12–23] by means of mathemati-
cal methods, developed in refs. [24–28] for the equivalent integrals over $N \times N$
matrices, using a $1/N$ expansion.

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An important example of a bosonic string, embedded in one space–time dimension, was investigated in the case of spherical topology of the world sheet [13], using an exact solution for the matrix anharmonic quantum oscillator in the large $N$ limit [24].

It was found in ref. [13] that the string susceptibility $\chi(\lambda)$ near the critical value $\lambda_c$ of the bare cosmological constant $\lambda$ reaches a continuum limit at which its critical behaviour is

$$\chi(\lambda) \approx \frac{-1}{\log(\lambda_c - \lambda)}$$

and for the mean square extent $\langle x^2 \rangle$ (in a single dimension)

$$\langle x^2 \rangle \approx \log^2(\lambda_c - \lambda).$$

It was also established that this model possesses infinitely many equidistant degenerate stable resonances [13].

Furthermore, the Green function $K(x)$ of two vertex operators of the form $e^{ipx}$, was calculated for this model in ref. [29]; it has a Fourier transform

$$K_p = \frac{1}{2\pi} \int dx \, e^{ipx} K(x) \approx \frac{m_0}{p} \tanh \frac{p}{m_0},$$

where $x$ is the distance between the two points in the one dimensional space, at which the vertex operators are located, and

$$m_0 \approx \frac{-1}{\log(\lambda_c - \lambda)}$$

is the mass gap; eq. (2) follows immediately from this result.

All three results appear to be in a perfect agreement with the KPZ approach [30] which predicts (see ref. [12] for definitions)

$$\gamma_{str} = 0, \quad \nu_{str} = 0,$$

a square logarithmic behaviour as in eq. (2) and even the large $p$ behaviour of $K_p$ (as noted in ref. [29]).

In ref. [31], the result (3) has been generalized to all multipoint correlation functions for a spherical topology of the world sheet.

This one dimensional bosonic string is interesting in many respects. First, it is equivalent to the model of a massless scalar free field coupled to 2D gravity, a model widely used as a starting point of many investigations and generalizations in conformal field theory. Second, it has a clear physical interpretation as a model of
2D quantum gravity developing in time (the role of which is played by a single bosonic field), or of a string living in a one dimensional space–time. One could discuss in principle questions such as the stability of the “2D universe” in physical time. Third, and probably the most important, is the possibility to investigate the string field theories on the boundary between “weak” and “strong” gravity, since the central charge of matter in this string model is \( c = 1 \), corresponding to 1D embedding. The strange logarithmic corrections in eqs. (1), (2) and (4) certainly reflect the influence of a phase transition at \( c = 1 \), whose nature remains the main puzzle of 2D gravity.

In this paper we shall investigate the field theory of closed strings in one physical (time) dimension in the double scaling limit \([1-5]\), in which \( N \) goes to infinity \((1/N) \) is the string interaction coupling), and \( \lambda \to \lambda_c \), fixing a function \( \xi \) of \( N \) and \( \lambda_c - \lambda \). In this case the scaling parameter \( \xi \) is nontrivial and it is not equal to some product \( N(\lambda_c - \lambda)^p \), unlike the cases of \( c < 1 \). This scaling parameter will be determined below explicitly.

In sect. 2, we shall recall the definition of the model and its equivalence to the \( N \times N \) matrix quantum anharmonic oscillator. Then we shall rederive the representation of the corresponding functional integral in terms of a system of \( N \) noninteracting fermionics oscillators in a nonlinear potential.

In sect. 3 the scaling limit

\[
N \to \infty, \quad \lambda \to \lambda_c, \quad \xi = \mu N \text{ fixed}
\]

in which \( \mu(\lambda_c - \lambda, N) \) is the Fermi level of a system of \( N \) fermions, will be analyzed. It will be shown that in the scaling limit the problem is related to the behaviour of the density of levels in the inverted quadratic potential \( V = -x^2 \). The quasiclassical approach, which is exact in this case, gives a parametric relation between the string susceptibility \( \chi \) and \( \lambda \) of the form

\[
\chi = \chi(\mu, N), \quad \lambda = \lambda(\mu, N).
\]  

In sect. 4, starting from the analysis of the topological \((1/N)\) expansion of eq. (7), we shall establish the structure of singularities of \( \chi_g(\lambda) \) for the fixed genera \( g = 0, 1, 2 \) and compare it with the direct WKB expansion.

Sect. 5 will be devoted to a discussion of nonperturbative effects with respect to the string coupling \( 1/N \) and to some conclusions.

2. String field theory in one dimension as a system of \( N \) quantum anharmonic fermionic oscillators

Following the definitions of refs. \([8-10]\), we shall define this model as a model of dynamically triangulated random surfaces with the partition function

\[
Z(\lambda, N) = \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} N^{-2g} \sum G^\infty_\epsilon \int dx_1 \ldots dx_n \exp \left( - \sum_{\langle ij \rangle \in G_n^\epsilon} L(x_i-x_j) \right).
\]
The internal sum runs over all possible Feynman graphs $G^{(n)}_g$ with $\varphi^3$ vertices, dual to the triangulation, made of $n$ vertices ($n$ plays the role of the invariant area of the two dimensional curved manifold represented by this graph); $\langle ij \rangle$ is the link of the graph connecting the $i$th and $j$th vertex, and the topology is characterized by a genus $g$; $(-\log \lambda)$ is the bare cosmological constant ($\lambda$ itself will be regarded as the measure of this cosmological constant in what follows), and $1/N$ is the string coupling constant, corresponding to the amplitude of one closed string splitting into two closed strings.

The usual choice for the lattice lagrangian $L(x_i - x_j)$ in the case of a discretized bosonic Polyakov string is

$$L_p(x_i - x_j) = (x_i - x_j)^2,$$

which corresponds in the continuum to

$$\mathcal{L}_p = g_{ab} \partial_a x \partial_b x.$$  \hspace{1cm} (10)

It was argued in ref. [13] that the replacement of (9) by

$$L(x_i - x_j) = |x_i - x_j|$$  \hspace{1cm} (11)

should not influence the critical properties of the model (as well as the change $\mathcal{L}_p \rightarrow \mathcal{L} = |g_{ab} \partial_a x \partial_b x|^{1/2}$), since it corresponds to the same universality class. This may be seen from the fact that the perturbative integrals in (8) are ultraviolet convergent; therefore the change of the action modifies only the short distance, nonuniversal, properties of the model.

We recall too that the standard duality transformation in (8) brings us to the same representation (8), but with the dual coordinates $p_i$ corresponding to the vertices of dual triangulations, with the lagrangian

$$L(p_i - p_j) = -\log \left[1 + (p_i - p_j)^2 \right].$$  \hspace{1cm} (12)

The use of nongaussian interactions (11) and (12) is the price to pay for the exact solvability of the theory (8). Indeed, as was pointed out in ref. [13], eq. (8) is equivalent to the Feynman graph expansion of the functional integral for the vacuum to vacuum transition amplitude in the time $T$ for the matrix anharmonic oscillator:

$$Z(T) = \langle 0|e^{-iH|0} = \int D^N \varphi(t) \exp \left\{-N \int_0^T dt \left[\varphi(t)^2 + V(\varphi)\right]\right\},$$  \hspace{1cm} (13)

in which for definiteness we shall work with $V(\varphi) = \varphi^2 + \lambda \varphi^3$, but it will be made
clear that it could be a more general function. In eq. (13) \( \varphi_{ij}(t) \) is a time dependent \( N \times N \) hermitian matrix. The role of the coordinates \( x_i \) of the vertices in eq. (8) will be played in the diagram technique given by eq. (13) by the physical times \( t_i \). On the other hand, \( t_i \) represents the lattice scalar field, defined at the vertices of a graph.

It has been shown in ref. [24], from the Schrödinger equation associated with the functional integral (13), that this problem may be reduced to the study of \( N \) independent fermions in an external potential. This result will be rederived here for completeness from the functional integral formalism.

Following ref. [24], we represent \( \varphi(t) \) as

\[
\varphi_{ij} = \sum_k \Omega_{ik} z_k \Omega_{kj},
\]

where \( z_k \) are the eigenvalues of the matrix \( \varphi \), and \( \Omega \) is the unitary matrix which diagonalizes \( \varphi \). This gives the functional Dyson measure

\[
D^{N^2} \varphi(t) = \prod_{t \in [0, T]} \prod_{i > j} \left[ z_i(t) - z_j(t) \right]^2 \prod_{i=1}^{N} Dz_i(t) D_{U(N)} \Omega,
\]

where \( D_{U(N)} \Omega \) is the U(N) Haar measure.

Further, we have

\[
\text{tr} \varphi^m(t) = \sum_{i=1}^{N} z_i^m
\]

and

\[
\text{tr} \varphi^2 = \text{tr} \left( \frac{d}{dt} (\Omega^* z \Omega) \right)^2 = \sum_{i=1}^{N} z_i^2 + \sum_{i,j=1}^{N} (z_i - z_j)^2 |A_{ij}|^2
\]

in which

\[
A_{ij} = - (\dot{\Omega} \Omega^{-1})_{ij}
\]

is the element of the algebra of the group U(N). The measure \( D_{U(N)} \Omega \) in eq. (15) can be replaced, due to the group invariance, by the linear measure \( DA_{ij}(t) \).

If the initial and final states are U(N) invariant one can integrate out the variables \( A_{ij} \):

\[
\prod_{t \in [0, T]} \left( \prod_{i > j} \left[ z_i(t) - z_j(t) \right]^2 \right) DA(t) \exp \left( -\frac{N}{2} \int_0^T dt \sum_{ij} (z_i - z_j)^2 |A_{ij}|^2 \right).
\]

At first sight, the gaussian integral over \( A \) gives a determinant, which cancels exactly the Van der Monde determinant from the measure, and expression (19) is
independent of the constants \( z_i \). But according to (17), (18) a correct regularization of the exponent in expression (19) is

\[
\sum \int_0^{T-\varepsilon} dt \left[ z_i(t) - z_j(t) \right] \left[ z_i(t+\varepsilon) - z_j(t+\varepsilon) \right] |A_{ij}(t)|^2.
\]

(20)

This representation (20) arises because the matrix \( \Omega_{ij}(t) \) is defined exactly at the point \( t \), but the connection \( A_{ij}(t) \) is defined on an infinitesimal neighbourhood of \( t \).

From expressions (19) and (20) we can see, that all Van der Monde determinants cancel, except for the first and the last ones (raised to the power one). Thus we get, instead of eq. (13),

\[
\hat{Z}(T) = \int \prod_{i=1}^{N} Dz_i(t) \Delta(z(0)) \Delta(z(t))
\]

\[
\times \exp \left( -N \int_0^T dt \sum_{i=1}^{N} \left( \dot{z}_i^2 + z_i^2 + \frac{1}{3} \dot{z}_i^3 \right) \right),
\]

(21)

where

\[
\Delta(z(t)) = \prod_{i>j} \left[ z_i(t) - z_j(t) \right]
\]

is the Van der Monde determinant.

The representation (21) shows that the time evolution is that of independent particles in a potential. The product \( \Delta(z(0))\Delta(z(T)) \) in eq. (21) leads to the antisymmetrization of the final states of the \( N \) oscillators with respect to the initial states, the signature of Fermi statistics. This derivation can be easily made more rigorous if one introduces a time lattice [28].

Finally, we can obtain the limit \( T \to \infty \) of eq. (13) by calculating the ground state energy \( E_0(N, \lambda) \) of \( N \) noninteracting fermions:

\[
\hat{Z}(T) \sim e^{-TE_0} \text{ for } T \text{ large.}
\]

(22)

This ground state is a Slater determinant made of the \( N \) lowest eigenstates of the Schrödinger operator

\[
\left( -\frac{1}{N^2} \frac{d^2}{dz^2} + z^2 + \frac{1}{3} \lambda z^3 - \varepsilon_k \right) \psi_k(z) = 0.
\]

(23)

The topological string interaction constant \( 1/N \) plays the role of the Planck constant \( h \). The potential in eq. (23) is unstable for quantum particles and it is only
in the classical limit $N \to \infty$ that we can recover discrete levels. This is just the limit in which all the formulae (1)-(5) for the spherical topology have been obtained.

We assume in the following that $N$ is not infinite, but that it is sufficiently large so that we can write the usual formula for the ground state of a system of noninteracting fermions:

$$E_0 = \sum_{i=1}^{N} \varepsilon_i$$

(24)

in which $\varepsilon_N$ is the Fermi level.

3. The singularity of the density of states in the scaling limit

The problem reduces to the investigation of the total energy of $N$ independent fermions, i.e. to the calculation of the density of states for the one particle Schrödinger eigenvalue problem

$$\left(-\frac{d^2}{dz^2} + U(z) - E_k\right)\psi_k(z) = 0$$

(25)

in a potential $U(z)$ which scales with $N$ as

$$U(z) = NU\left(zN^{-1/2}, g_i\right).$$

(26)

The $g_i$ are the coupling constants which characterize this potential; they remain finite in the limit which is studied below. The particular form (23) of the potential $U$ corresponds to a summation over surfaces made of triangles, which are dual to $\varphi^3$ graphs, and it has a single coupling constant $\lambda$.

In the large $N$ limit the potential becomes large; the characteristic length scale is of order $(N)^{1/2}$ and the energy scale is of order $N$. Therefore the problem becomes quasiclassical; the spectrum of eigenvalues becomes dense in this limit and the density of levels is nonsingular everywhere except at the points $E_c$ at which $U'(z_0) = 0$. Here $z_0$ is a classical turning point defined by $U(z_0) = E_c$.

Near $E_c$ the density of states is singular in our scale and it is this singularity that controls the critical behaviour of the random lattice problem.

The most generic type of singularity corresponds to a parabolic maximum of the potential. Suppose that near $z = z_0$ the potential behaves as

$$U(z) = E_c - \frac{1}{2}(z - z_0)^2 + N \sum_{k=3}^{\infty} \frac{\tilde{g}_k(z - z_0)^k}{N^{k/2}}.$$  

(27)

Here $\tilde{g}_k$ are finite nonsingular coefficients which depend upon the coupling
constants \( g_i \). A finite rescaling of \( z \) can always be made to set \( \bar{g}_2 \) equal to \(-\frac{1}{4}\), as in eq. (27), provided it is nonzero. We could consider points of higher criticality if, at the turning point, the coefficients \( g_i \) were tuned in order to get \( \bar{g}_k = 0 \) for \( k = 1, 2, \ldots, m \). Here we limit ourselves to the generic type of singularity at which \( \bar{g}_2 \) does not vanish.

It is convenient to shift the maximum of the potential to the origin by substituting \( x \) to \( z - z_0 \) and to define \( \xi = E_c - E \) as the new energy variable. The eigenvalue problem (25) takes the form

\[
\left( -\frac{d^2}{dx^2} - \frac{1}{4}x^2 + \frac{\bar{g}_3}{\sqrt{N}} x^3 + \ldots + \xi_k \right) \psi_k = 0, \tag{28}
\]

in which the constant \( \bar{g}_3 \) is proportional to the cosmological constant \( \Lambda \) introduced above.

The main observation, from which the analysis of the singularity near \( \xi = 0 \) becomes simple, is that it is essentially controlled by the quadratic part of the potential. The potential of eq. (23) has the form of a deep well with an impenetrable wall to the left and a potential barrier to the right. It is not a loss of generality to keep in mind a picture of this kind in eq. (28). Actually there is a characteristic scale of \( x \sim \Lambda \) at which the particular form of the potential (28) becomes important. The cubic term gives the estimation \( \Lambda \sim \sqrt{N/\bar{g}_3} \). The corresponding characteristic energy scale is \( E_0 \sim \Lambda^2 \). On the other hand the region of the singularity corresponds to \( \xi \) small in that scale. Nevertheless, \( \xi \) should be held large (of order \( N \) actually) in order to remain in the vicinity of the singular point for the density of levels. Therefore if \( 1 \ll \xi \ll \Lambda^2 \) we have a large coordinate range

\[
(\xi)^{1/2} \ll x \ll \Lambda, \tag{29}
\]

in which we can neglect the higher order terms in the potential, as well as apply the quasiclassical approximation. Let us call \( \psi_{R}(x, \xi) \) the unique solution of eq. (28) which satisfies the “right” boundary condition. In the region (29) it has the form [32]

\[
\psi_{R}(x) = \frac{1}{\sqrt{x}} \sin \left[ \frac{1}{4} x^2 + \xi \log x/\Lambda + \frac{1}{2} \Phi(\xi/E_0, g_i) \right], \tag{30}
\]

in which \( \Phi(\xi/E_0, g_i) \) is a nonuniversal phase which depends upon the specific form of the potential that we have chosen. Since \( \xi \) is small compared to the overall energy scale of the potential and since the phase is regular at the critical point, \( \xi \) can be consistently set to zero in expression (30) and we have a single constant \( \Phi(0, g_i) \) in which the nonuniversal information about the potential is concentrated. On the other hand in the region \( x \ll \Lambda \) we can solve eq. (28) exactly since we keep
only the parabolic term in the potential. In the region (29) the asymptotic form of
the solution is

\[ \psi_L(x) \approx \frac{1}{\sqrt{x}} \sin \left[ \frac{1}{2} x^2 + \xi \log x + \frac{1}{2} \phi(\xi) \right], \]  

(31)

where the “left” phase \( \phi(\xi) \) is to be matched with that of eq. (30). It is determined
up to terms which are exponentially small in \( \xi \):

\[ \phi(\xi) = -\frac{1}{2} i \log \frac{\Gamma\left(\frac{1}{2} + i\xi\right)}{\Gamma\left(\frac{1}{2} - i\xi\right)} + O(e^{-2\pi \xi}). \]  

(32)

The exponentially small terms in \( \phi(\xi) \) correspond obviously to tunneling through
the parabolic barrier and therefore to a nonuniversal dependence on the specific
boundary conditions at the left hand side of the barrier.

Comparing eqs. (30) and (32) one finds the quantization condition

\[ \Phi_0 - \xi \log L - \phi(\xi) = \pi n \]  

(33)

from which the density of levels follows:

\[ \rho(\xi) = -\frac{1}{\pi} \left( \frac{\partial \phi(\xi)}{\partial \xi} + \log L \right). \]  

(34)

With the explicit formula (32) we are in a position to develop the topological
expansion in powers of \( 1/N^2 \). Since \( \xi \) is of order \( N \) (which is small compared to
the potential energy scale \( E_0 \)), this is simply the asymptotic expansion for large \( \varepsilon \) of
the \( \Gamma \) functions in eq. (32). We thus find, up to a \( \xi \) and \( N \) independent constant:

\[ \rho = -\frac{1}{4\pi} \left[ \psi\left(\frac{1}{2} + i\xi\right) + \psi\left(\frac{1}{2} - i\xi\right) \right] + \frac{1}{\pi} \log \Lambda \approx -\frac{1}{2\pi} \log \mu + \frac{1}{2\pi} \sum_{g=1}^{\infty} \frac{C_g}{\xi^{2g}}. \]  

(35)

\( \psi \) is the dilogarithm function, and the coefficients \( C_g \) of the topological expansion
are related to the Bernoulli numbers by

\[ C_g = (-1)^{g+1} \frac{B_{2g}}{2g} (1 - 2^{1-2g}). \]  

(36)

4. Topological expansion: internal and external fractal properties

The direct expansion in powers of the string coupling constant \( 1/N^2 \), may be
recovered from the WKB expansion. At leading order (genus zero) this was done
in ref. [24]; at next order (torus) this was calculated in ref. [33]. A systematic expansion in powers of $1/N^2$ may be derived from a systematic WKB expansion. This may be done for instance by the following method: the Schrödinger operator for the pseudo-fermions reads

$$-rac{1}{N^2} \frac{\delta^2}{\delta x^2} + V(x) - \varepsilon \phi(x) = 0$$

and $1/N$ plays the role of Planck’s constant $\hbar$.

From the density of eigenvalues

$$\rho(\varepsilon) = \sum \delta(\varepsilon - \varepsilon) = \text{tr} \delta(\varepsilon - H),$$

one defines the Fermi energy $\mu$ as

$$N = \int^{\mu} d\varepsilon \rho(\varepsilon)$$

and the free energy of the 1D matrix model is given as

$$F = \frac{E}{N^2} = \frac{1}{N^2} \int^{\mu} d\varepsilon \varepsilon \rho(\varepsilon).$$

We want to expand systematically $\rho(\varepsilon)$ in powers of $1/N$ and this may be done from the representation

$$\rho(\varepsilon) = \text{Im} \int_0^\infty \frac{dt}{\pi} e^{it\varepsilon} \text{tr}(e^{-itH})$$

and

$$e^{-itH} = e^{-itp^2} e^{-itV} A.$$
the operator $A$ by ordering all $p$’s on the left and all $x$’s on the right:

$$A(x, p) = 1 + \frac{1}{2}t^2 \left( -\frac{2ip}{N} V' + \frac{V''}{N^2} \right) + \frac{it^3}{3} \left( \frac{8}{N^2} p^2 V'' - \frac{2V'^2}{N^2} \right)$$

$$- \frac{t^4}{2N^2} V''^2 p^2 + O\left( \frac{1}{N^3} \right).$$

This gives

$$\rho(\varepsilon) = \frac{N}{2\pi} \left[ \int \frac{dx}{p(x)} - \frac{1}{2N^2} \frac{\partial^2}{\partial \varepsilon^2} \int \frac{V''}{p(x)} \, dx - \frac{1}{3N^2} \frac{\partial^3}{\partial \varepsilon^3} \int \frac{4p^2(x) V'' - V'^2}{p(x)} \, dx \right.$$  

$$\left. - \frac{1}{2N^2} \frac{\partial^4}{\partial \varepsilon^4} \int \frac{V''^2 p^2}{p(x)} \, dx + O\left( \frac{1}{N^4} \right) \right],$$

(44)

in which $p^2(x) = \varepsilon - V(x)$ and the integrals are taken between two turning points; this expression may be considerably simplified by straightforward algebraic manipulations.

Let us apply this to the potential $V(x) = x^2 + \frac{1}{3} x^3$ of our problem, but before we shift and rescale $x$ to $-2/\lambda + 2^{-1/2} x$, so that the Schrödinger equation becomes

$$\left(- \frac{1}{N^2} \frac{d^2}{dx^2} - \frac{1}{4} x^2 + \lambda x^3 - \mu \right) \varphi(x) = 0$$

(45)

(in which $\lambda/12\sqrt{2}$ has been renamed $\lambda$) and

$$\mu = \left( \frac{1}{3} \varepsilon - \frac{2}{3\lambda^2} \right) \quad (\mu < 0).$$

(46)

Near the critical value of $\lambda$ the Fermi energy in the shifted energy scale is close to zero and therefore we have to study only the vicinity of $\mu = 0$.

The density of eigenvalues $\rho(\mu)$ of the Schrödinger operator $H = p^2 + \tilde{V}(x)$, $\tilde{V}(x) = -\frac{1}{2} x^2 + \tilde{\lambda} x^3$ is expanded as above and we obtain from eq. (44) with $p(x) = (\mu + \frac{1}{2} x^2 - \tilde{\lambda} x^3)^{1/2}$ an expansion of $\rho$ in powers of $1/N^2$, which near the singular point $\mu = 0$ of the spectrum, reads, as first obtained at this order in ref. [33],

$$\rho(\mu) = \frac{N}{2\pi} \left[ -\log(-\mu) + \frac{1}{24N^2 \mu^2} + O\left( \frac{1}{N^4} \right) \right].$$

(47)
Up to a rescaling of \( \rho \), the expression (47) is of the form

\[
\rho(\mu) = -\log \frac{\mu}{A} + \frac{A}{N^2\mu^2} + \frac{B}{N^4\mu^4} + \ldots ,
\]

(48)
in which \( A = \frac{1}{24}, B = \frac{7}{960} \).

In eq. (48) the parameter \( \mu \) depends implicitly on the original cosmological constant \( \lambda \). In order to obtain this dependence we have to remember that if we rescale \( x \) by \( 1/\lambda^2 \) the density \( \rho(\mu) \) is normalized in such a way, that

\[
\int_{\mu_0}^{\mu} d\nu \rho(\nu) = \lambda^2 .
\]

(49)

This gives us the implicit dependence of \( \mu(\lambda) \). Inserting eq. (48) in eq. (49), we obtain

\[
\lambda_c - \lambda = -\mu \left( \log \frac{\mu}{A} - 1 \right) - \frac{A}{N^2\mu} - \frac{B}{3N^4\mu^3} - \ldots .
\]

(50)

To get the explicit topological expansion of \( \rho(\lambda) \), we have to eliminate \( \mu \) in between eq. (48) and eq. (50) order by order in \( 1/N \).

The density of states \( \rho(\lambda, N) \) is related to another interesting physical quantity, namely the two point function

\[
\chi(\lambda, N) = \frac{\partial^2 E(\lambda, N)}{\partial \lambda^2} ,
\]

(51)

where \( E(\lambda, N) \) corresponds to the free energy of our model, or to the ground state of \( N \) fermions. Indeed, since

\[
\lambda^2 E(\lambda, N) = \int_{\mu_0}^{\mu} d\nu \nu \rho(\nu)
\]

(52)
from eqs. (49)-(52) we obtain

\[
\lambda_c^{-1} \chi(\lambda, N) = -\frac{1}{\rho(\lambda, N)} .
\]

(53)

Finally, from eqs. (48) and (50), we find the following topological expansion for eq. (53):

\[
\chi(\lambda, N) = \chi_0 + \frac{1}{N^2} \chi_1 + \frac{1}{N^4} \chi_2 + \ldots ,
\]

(54)
with the two point functions for genus zero

\[ \chi_0(\lambda) = \frac{1}{L}, \quad L = \log \frac{\mu_0(\lambda)}{A}, \]  \hspace{1cm} (55)

for genus one

\[ \chi_1(\lambda) = \frac{1}{24\mu_0^2(\lambda)} \left( \frac{1}{L^2} + \frac{1}{L^3} \right), \] \hspace{1cm} (56)

and for genus two

\[ \chi_2(\lambda) = \frac{1}{\mu_0^4(\lambda)} \left[ \frac{7}{960} \left( \frac{1}{L^2} + \frac{1}{L^3} \right) - \frac{1}{576} \left( \frac{3}{L^3} + \frac{5}{2L^4} + \frac{1}{2L^5} \right) \right], \] \hspace{1cm} (57)

where \( \mu_0(\lambda) \) is defined by the transcendental equation

\[ \mu_0(\lambda) \left( \log \frac{\mu_0(\lambda)}{A} - 1 \right) = (\lambda_c - \lambda). \] \hspace{1cm} (58)

We see from eqs. (54)–(58), that the dependence on the cosmological constant, although it is universal, is rather complicated for this \( c = 1 \) (or \( d = 1 \)) matter interacting with 2D gravity. It is not given by a simple power-like singularity, like in the \( c < 1 \) models [1–5].

In the limit \( \lambda_c - \lambda \ll A \) we obtain from eq. (58), up to \( \log \log \) corrections,

\[ \mu_0(\lambda) \approx \frac{\lambda_c - \lambda}{-\log[(\lambda_c - \lambda)/A]}, \] \hspace{1cm} (59)

and the formulae (55)–(57) simplify to

\[ \chi_0(\lambda) = \frac{1}{\log[(\lambda_c - \lambda)/A]}, \] \hspace{1cm} (60)

which coincides with eq. (48),

\[ \chi_1(\lambda) = \frac{1}{24(\lambda_c - \lambda)^2}, \] \hspace{1cm} (61)

\[ \chi_2(\lambda) = \frac{7 \log^2[(\lambda_c - \lambda)/A]}{960(\lambda_c - \lambda)^4}. \] \hspace{1cm} (62)

The “naive” KPZ–DDK [30, 34, 35] scaling

\[ \gamma_{\text{str}} = 2 - \gamma, \] \hspace{1cm} (63)
which would lead in eqs. (60) and (62) to powers of $\lambda_c - \lambda$, is violated by powers of logarithms of $\lambda_c - \lambda$.

Let us conclude this section with a discussion on the possible physical interpretation of these results. In the scaling limit, the physical mass spectrum of our theory is nonequidistant, unlike the case of spherical topology [13]. It is defined by the one-particle density of levels $\rho(\lambda, N)$, given by eqs. (35) and (49). The energies of physical resonances correspond to the excitation energies of a system of noninteracting fermions. In particular, the mass gap $m_0(\lambda, N)$ is now defined by

$$m_0 = \frac{\lambda_c^2}{N \rho(\lambda, N)} \approx \frac{\lambda_c^2}{N \log \mu} \left(1 - \frac{1}{24 \log \mu} + \ldots\right). \quad (64)$$

For the spherical topology, up to logarithmic corrections, $\mu$ is proportional to $\lambda - \lambda_c$ and thus to $(A_{int})^{-1}$ if we call $A_{int}$ the characteristic area of the manifold. On the other hand $m_0$ is proportional to $(L_{ext})^{-1}$, where $L_{ext}$ is the characteristic size of the system in the external one dimensional embedding space (in euclidean picture), or the characteristic physical time.

We obtain from expression (64) a relation between $A_{int}$ and $L_{ext}$ of the form

$$A_{int} \approx e^{c L_{ext}}, \quad (65)$$

which is a reflection of the connexion between the internal and external fractal properties of quantum geometry in our model. A relation similar to expression (65) holds (up to pre-exponential factors) for any genus.

5. Discussion

The model considered here is the first known example of a system of 2D gravity coupled to a matter field that exhibits a scaling behaviour in terms of a nontrivial scaling parameter which does not reduce to a cosmological constant $\lambda_c - \lambda$ multiplied by a power of the topological constant $1/N$. As the expansion in $1/N$ shows, the simple power like scaling, which is predicted by KPZ–DDK theory [30, 34, 35], is violated by logarithmic corrections. For this model, such corrections are universal, and one has to consider them seriously if one hopes to reach a physically acceptable scaling limit.

Obviously, the $c = 1$ matter is an exceptional case already for conformal field theories in flat space since it possesses well-known nonuniversal features: a marginal operator appears in $c = 1$ models, leading to a dependence of the critical exponents on a parameter (the radius of compactification for of a free field).

This circumstance should be important in the presence of gravity as well. We see the origin of the violation of KPZ scaling in this lack of universality of the $c = 1$ matter. One can speculate that we have considered the case of an infinite radius of
compactification of a free bosonic field (corresponding to the “time” embedding of 2D gravity), and the case of finite radius could give, in general, some other corrections to the simple scaling.

The idea of a nontrivial scaling parameter in the double scaling limit \((N \to \infty, \lambda_c - \lambda \to 0)\) could be a good starting point for getting some insight into the “real physics”, namely matter with \(c > 1\). Our model is a boundary case, which could a priori belong already to the \(c > 1\) phase. At least, it is the only solved model of field theory of strings, which has an infinite number of stable resonances.

An interesting point is the appearance of nonperturbative effects in this model, which are reminiscent of the corresponding phenomenon for \(c < 1\) models \([1-5]\), but they have here a transparent physical interpretation: the 2D gravity is unstable in physical time. Indeed, the inverted quadratic potential that we have considered did not have stable states; the energy levels that we considered had in fact imaginary parts related to the transparency of the barrier. Near the Fermi level this adds a nonnegligible imaginary part to our result. Therefore the result presented above is simply a book-keeping of the topological expansion, but we do not understand how to fix the nonperturbative arbitrariness. This means presumably, that our quantum lattice manifold will expand exponentially slowly in time (with an amplitude proportional to \(\exp(-2\pi N\mu)\) for sufficiently large \(N\mu\)). Furthermore if we consider a physical quantity such as the string susceptibility \(\chi = -1/\rho\), we obtain the following topological expansion:

\[
\frac{1}{2\pi} \chi = -\frac{1}{\log(1/\mu)} + \frac{1}{\log(1/\mu)} \sum_{g=1}^{\infty} \frac{1}{\varepsilon^{2g}} \left[ \frac{C_g}{\log(1/\mu)} + O\left(\frac{1}{[\log(1/\mu)]^2}\right) \right].
\]

This expansion is asymptotic and for large genuses the coefficients \(C_g\) grow like \((2g)!\):

\[
C_g \approx \frac{(2g - 1)!}{(2\pi)^{2g-2}}.
\]

All the coefficients in front of the leading terms in \([\log(1/\mu)]^{-1}\) are positive (except for the first one corresponding to a spherical topology), as it should be in a unitary theory. This shows that the topological expansion for this model as well, is non-Borel summable and therefore has necessarily some nonperturbative ambiguities.

We have benefitted from stimulating discussions with I. Kostov, A. Schwimmer and S. Shenker. Al. Zamolodchikov thanks the Physics Department of the ENS for its hospitality.
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