Integrable deformations of the $O(3)$ sigma model.
The sausage model

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We consider two one-parameter families of two-dimensional relativistic factorized scattering theories, which are deformations of the ones characteristic for the two-dimensional $O(3)$ sigma models with $\theta = 0$ and $\theta = \pi$. The Bethe ansatz technique is applied to these two families to justify their interpretation as the factorized scattering theories of certain $O(2)$-symmetric deformations of the $O(3)$ sigma model (the sausage model). The result suggests that the sausage model is integrable at the topological angle values $\theta = 0$ or $\theta = \pi$.

1. Introduction

The non-linear sigma models in two-dimensional space-time are widely discussed in field theory as continuous models of two-dimensional spin systems (see e.g. refs. [1–4] and references therein) as well as in relation to string theory (e.g. refs. [5–9]). The general two-dimensional sigma model (SM) is defined through the action

$$A[G] = \frac{1}{2} \int G_{ij}(X) \partial_\mu X^i \partial_\mu X^j d^2x + \ldots,$$

(1.1)

where coordinates $x^\mu, \mu = 1, 2$ span a two-dimensional flat space-time, while the fields $X^i, i = 1, 2, \ldots, d$ are coordinates in a $d$-dimensional Riemann manifold.

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\( \mathcal{M} \) called the target space. The symmetric matrix \( G_{ij}(X) \) is the corresponding metric tensor.

The standard approach to (1.1) is perturbation theory. If the curvature of \( G_{ij} \) is small the action (1.1) is perturbatively renormalizable and one can use the following one-loop renormalization group (RG) evolution equation [3]:

\[
\frac{d}{dt} G_{ij} = -\frac{1}{2\pi} R_{ij} + O(R^2),
\]

(1.2a)

where \( t \) is the RG “time” (the logarithm of scale) and \( R_{ij} \) is the Ricci tensor of \( G \).

Qualitative considerations give the impression that in general the non-linear evolution equation (1.2) is unstable in the sense that even if one starts with a manifold of small curvature everywhere at some scale \( t_0 \), underevolution in both directions \( t \to \pm \infty \) the metric \( G(t) \) develops at least some regions where its curvature grows and (1.2) is no more applicable. If it happens in the ultra-violet (UV) direction \( t \to -\infty \) the action (1.1) does not define any local field theory and can be considered at most as an effective theory for scales around \( t_0 \). However, special solutions exist where the UV direction \( t \to -\infty \) is stable and the curvature remains small everywhere up to \( t = -\infty \), permitting one to define a local field theory (at least perturbatively). For example, if \( \mathcal{M} \) is a compact Einstein manifold, its metric grows homogeneously as \( t \to -\infty \), the curvature monotonously decreases and we are dealing with an UV asymptotically free field theory unambiguously defined by the action (1.1). In these UV-stable situations the short-distance behavior can be studied perturbatively and is determined mainly by the RG flow (1.2) of the metric. On the contrary, at large distances the metric typically flows to configurations with growing curvature and perturbation theory fails to recognize, even qualitatively, the most interesting physics, i.e. the spectrum of excitations and their scattering. Moreover, the large-distance pattern depends strongly on other possible terms in the action (1.1), like tachyon, Wess–Zumino–Witten or topological terms (which are omitted in eq. (1.1) since they are less important in the UV region).

To study this large-distance physics one has to find a suitable non-perturbative approach. For example, isometries of \( G_{ij} \), if any, manifest themselves as global symmetries of the SM. The corresponding scattering theory should respect these symmetries. Obviously, symmetries of this kind are not enough even to predict qualitative characteristics of the infrared (IR) behavior (mass gap, etc.). Critical situations, which correspond to fixed points of (1.2), are much more symmetric and subject to powerful methods of conformal field theory (CFT) [10]. This permits one to analyse even non-perturbative fixed-point manifolds of large curvature [11,12].

Quantum integrability is one of the most successful lines in studying the off-critical SM’s. There is a number of trajectories of (1.2) which are known (or
conjectured to be) integrable, i.e. the corresponding field theories possess infinite series of higher integrals of motion [13,15–17]. Examples of integrable flows are the $d$-dimensional spheres [13–15] and the semi-simple Lie group manifolds with invariant metric [18,19].

Typically it is hard to recognize integrable SM’s by just looking at the action (1.1). Even if one finds an action with classically integrable equations of motion, the quantum integrability is difficult to prove (and by no means assured). On the other hand, if a quantum SM is integrable, the higher integrals of motion are apparently manifested in the factorized scattering theory (FST) of the corresponding relativistic excitations. The FST itself is rather rigid and in fact its internal restrictions do not permit a very wide variety of consistent constructions.

It seems therefore that the following “inverse scattering” program may turn more successful. One constructs (or selects from known explicit constructions) an FST which looks suitable for a SM interpretation. In principle, an FST contains all the information about the background integrable field theory. Several techniques (the form factor bootstrap [22,23] or the Bethe ansatz methods [18–21,24–27]) allows one to compute some off-mass-shell observables on the basis of FST. In the UV region these observables are compared with the perturbative expansions generated by the SM action (1.1). If they match non-trivially in the UV limit, one has an indirect evidence of the SM integrability and it is natural to suggest the FST chosen as the scattering theory of the integrable SM. Moreover, one could use the FST as a non-perturbative definition of the SM.

Apparently this program implies a lot of guesswork. A more ambitious “inverse scattering” program would be to reconstruct directly an exact trajectory $G_{ij}(t)$ on the basis of the FST data.

In the body of the paper we argue in this manner a new integrable SM where the target space $M$ is the two-dimensional sphere $S^2$ ($d = 2$). Below we always imply that $M$ has this topology. Few observations about this particular kind of $M$ are in order.

The general action (1.1) (without tachyon) reads in this case

$$\mathcal{A}_{\theta}[G] = \frac{1}{2} \int G_{ij}(X) \partial_\mu X^i \partial_\mu X^j + i\theta T,$$

(1.3)

where $X^i$, $i = 1, 2$ are coordinates on $S^2$. The $S^2$ topology of the target space admits topologically non-trivial configurations of the fields $X^i(x)$ called the instanton configurations [28]. Any configuration can be viewed as a map $S^2 \to S^2$ from the (compactified) two-dimensional euclidean space-time to the target space. Topologically this map is completely characterized by the integer-valued degree $T$, which is called the instanton charge and in general should be added to the action (1.3). The “topological angle” $0 \leq \theta < 2\pi$ is an important additional parameter of the SM (1.3). From the path integral point of view every observable
\( O(\theta) \) can be considered as a sum
\[
O(\theta) = \sum_{q \in \mathbb{Z}} e^{iq\theta} O_q
\]
over the "q-instanton" contributions \( O_q \). While the topological term in the action (1.3) does not influence essentially the UV behavior (in particular, the perturbative expansion is completely independent of \( \theta \)), the instanton contributions are important at large distances. The on-mass-shell physics is expected to be strongly \( \theta \)-dependent.

At \( d = 2 \) the RG equation (1.2) is much simplified. First, \( R_{ij} = \frac{1}{2} R \delta_{ij} \), where \( R \) is the scalar curvature. Then one always can choose (at least locally) the conformal coordinates \( X^i \) on \( \mathcal{M} \) so that
\[
G_{ij} = e^{\Phi} \delta_{ij}
\]
with a single function \( \Phi(X) \). The RG evolution equation now reads
\[
- \frac{d\Phi}{dt} = \frac{1}{4\pi} R + \frac{1}{(4\pi)^2} R^2 + \ldots ,
\]
where the two-loop correction to (1.2) is also included [3]. Keeping only the (one-loop) term linear in \( R \) we have
\[
\frac{d}{dt} e^\Phi = \frac{1}{4\pi} \left( \frac{\partial}{\partial X^i} \right)^2 \Phi .
\]

It seems that this interesting non-linear partial differential equation is not much studied. In the UV direction \( t \to -\infty \) its solutions are expected to be in general unstable (however, in sect. 3 we present a one-parameter family of stable solutions) and to lead to a constantly growing manifold. For, define the volume of \( \mathcal{M} \),
\[
V = \int \sqrt{G} d^2 X .
\]
From (1.7) one finds that \( dV/dt = -2 \), so that
\[
V(t) = -2(t - t_0)
\]
and \( V \) grows linearly for \( t \to -\infty \). Contrary, the "forward" RG evolution always ends up at some "time" \( t_0 \), where \( \mathcal{M} \) shrinks to a point. This means that the large-distance behavior of SM (1.3) is always non-perturbative. At some scale near the "logarithmic pole" \( t_0 \) the target space \( \mathcal{M} \) becomes very small, the curvature becoming necessarily large.

The most studied example of SM on \( S^2 \) is the \( O(3) \) sigma model, which corresponds to the constant positive curvature metric on \( \mathcal{M} \) [1]. This model is commonly formulated as a field theory of an \( O(3) \) unit-vector (spin) field \( n_a(x) \), \( a = 1, 2, 3 \) (\( \sum_a n_a^2 = 1 \)) with the action
\[
A_{O(3)} = \frac{1}{2g} \sum_{a=1}^{3} \int (\partial_a n_a)^2 d^2 x + i\theta T .
\]
In terms of \( n_a(x) \) the instanton charge \( T \) can be written explicitly as follows:

\[
T = \frac{1}{8\pi} \int \sum_{abc} e^{abc} n_a \partial_{\mu} n_b \partial_{\nu} n_c \epsilon_{\mu\nu} \, d^2 x.
\]  

(1.11)

For reasons which will become clear later we call the O(3) SM with topological angle \( \theta \) the SSM\(_0^{(\theta)} \) field theory. The renormalization group in the O(3)-symmetric case becomes one-parameter and asymptotically free in the only coupling constant \( g \) (this coupling corresponds essentially to the constant scalar curvature \( R = 2g \)) which decreases logarithmically in the UV region,

\[
g \sim -\frac{2\pi}{\lambda}.
\]  

(1.12)

The standard asymptotically free perturbation theory in \( g \) is completely independent on \( \theta \). In particular, the UV conformal central charge is always \( c_{\text{UV}} = d = 2 \). In the observables (1.4) the \( \theta \)-dependence shows up in the form of non-perturbative instanton contributions [29,30]

\[
O_\theta \sim e^{-4\pi |q|/g_0},
\]  

(1.13)

where \( g_0 \) is the “bare” coupling, i.e. \( g \) at some small (UV-cutoff) scale \( r_0 \).

For general \( \theta \) the on-mass-shell pattern of the O(3) sigma model is unknown even qualitatively. Two particular points \( \theta = 0 \) and \( \theta = \pi \) are studied much better because at these points SSM\(_0^{(0)} \) is integrable [14,17,31]. Note that two integrable field theories SSM\(_0^{(0)} \) and SSM\(_0^{(\pi)} \) develop essentially different physics at large distances. While at \( \theta = 0 \) the correlation length is finite and all the excitations are massive, at \( \theta = \pi \) one observes a scale-invariant behavior in the IR limit (infinite correlation length). The large-distance asymptotics of SSM\(_0^{(\pi)} \) is described by a particular CFT. This CFT is in fact the SU(2) \( \times \) SU(2) WZW theory at level \( k = 1 \) [11] and below we denote it as SU(2)\(_1\). In addition to conformal symmetry (with central charge \( c = 1 \)) the SU(2)\(_1\) is invariant under right and left SU(2) current algebras generated by holomorphic and antiholomorphic currents \( J_a \) and \( \overline{J}_a \) (here \( a = 1, 2, 3 \) is the adjoint SU(2) index). From the RG point of view the SSM\(_0^{(\pi)} \) model can be considered as an interpolating trajectory which ends up at the IR fixed point characterized by SU(2)\(_1\) CFT (fig. 1). The incoming direction of the trajectory is determined mainly by the marginal operator \( \sum_a J_a \overline{J}_a \). In other words, the following expression:

\[
\mathcal{A}_f = \mathcal{A}_{\text{SU(2)}_1} + \frac{f}{2\pi} \int \sum_a J_a \overline{J}_a \, d^2 x
\]  

(1.14)

can be considered as an IR effective action of the O(3) SM at \( \theta = \pi \). Here the formal action of SU(2)\(_1\), \( \mathcal{A}_{\text{SU(2)}_1} \), is perturbed with some negative coupling \( f < 0 \). This sign of \( f \) corresponds to the zero-charge RG behavior and \( f \) goes to zero

\[
f \sim -\frac{1}{t}
\]  

(1.15)
for large distances $t \to \infty$. Action (1.14) can be used to generate a perturbative series in $f$ for the deviation of the IR behavior from that prescribed by the IR CFT. Note that the perturbation in (1.14) preserves the global diagonal $SU(2)$ symmetry, which actually reflects the original $O(3)$ symmetry of the action (1.10).

An integrable relativistic field theory is completely characterized by its FST. We recall here the FST's of the integrable $O(3)$ SM's at $\theta = 0$ and $\theta = \pi$.

At $\theta = 0$ (the $SSM_0^{(0)}$ field theory) the spectrum consists of a single $O(3)$ triplet of massive particles $A_a$, $a = 1, 2, 3$, with a non-perturbatively generated mass $m \sim r_0^{-1} e^{-2\pi/\sigma_0}$. The on-mass-shell momenta $(e, p)$ of massive particles are conveniently parameterized in terms of rapidities $-\infty < \beta < \infty$

$$e = m \cosh \beta, \quad p = m \sinh \beta.$$  

(1.16)

The factorized scattering respects the $O(3)$ isotopic symmetry and is completely defined by the two-particle amplitudes $S_{ab}^{cd}(\beta)$, where $\beta = \beta_1 - \beta_2$ is the rapidity difference of two colliding particles. It is convenient to describe FST in terms of the associative algebra of non-commuting symbols $A_a(\beta)$ [16] corresponding to the particles in the spectrum. In this formalism the two-particle amplitude appears in the commutation relation

$$A_a(\beta_1) A_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2) A_d(\beta_2) A_c(\beta_1).$$  

(1.17)

In our $O(3)$ symmetric case

$$S_{ab}^{cd}(\beta) = S_0(\beta)(P_0)_{ab}^{cd} + S_1(\beta)(P_1)_{ab}^{cd} + S_2(\beta)(P_2)_{ab}^{cd},$$  

(1.18)

where

$$(P_0)_{ab}^{cd} = \frac{1}{3} \delta_{ab} \delta_{cd},$$  

$$(P_1)_{ab}^{cd} = \frac{1}{3} \delta_{ac} \delta_{bd} - \frac{1}{3} \delta_{ad} \delta_{bc},$$  

$$(P_2)_{ab}^{cd} = \frac{1}{3} \delta_{ac} \delta_{bd} + \frac{1}{3} \delta_{ad} \delta_{bc} - \frac{1}{3} \delta_{ab} \delta_{cd}$$  

(1.19)

are the projectors on the two-particle states with isospin 0, 1 and 2, respectively.
The corresponding partial amplitudes are

\[ S_0(\beta) = \frac{\beta + 2i\pi}{\beta - 2i\pi}, \]
\[ S_1(\beta) = \frac{(\beta - i\pi)(\beta + 2i\pi)}{(\beta + i\pi)(\beta - 2i\pi)}, \]
\[ S_2(\beta) = \frac{\beta - i\pi}{\beta + i\pi}. \] (1.20)

The FST described has been subjected to Bethe ansatz analyses in refs. [21,31]. The results (although not very detailed) in the UV limit agree with the perturbative predictions for the O(3) SM. In particular, the UV conformal central charge \( c_{UV} \) is demonstrated to be 2 as expected.

The FST of SSM\(^{(\pi)}\) was suggested in ref. [17]. Since this field theory develops infinite correlation length (no mass gap), it is natural that its spectrum contains massless particles. These are the right- and left-moving SU(2)-doublets \( R_\sigma \) and \( L_\sigma \), where \( \sigma = \pm \) is the SU(2) isotopic index. As before we introduce the corresponding non-commuting symbols \( R_\sigma(\beta) \) and \( L_\sigma(\beta) \) to represent their factorized scattering in the algebraic form. The rapidity variable \( \beta \) is again used to parameterize the on-mass-shell momenta \((e,p)\), but in the massless case we have instead of (1.16)

\[ e = +p = \frac{M}{2}e^\beta \quad \text{for right-movers,} \]
\[ e = -p = \frac{M}{2}e^{-\beta} \quad \text{for left-movers,} \] (1.21)

so that the two-particle amplitudes depend on the rapidity difference only. The mass parameter \( M \) characterizes in fact the intercept scale \( M \sim r_0^{-1}e^{-2\pi/g_0} \) where the asymptotically free (Goldstone) UV behavior of SSM\(^{(\pi)}\) changes into the SU(2)-governed scale-invariant large-distance pattern. Note, that \( M \) becomes precisely specified after some non-trivial right–left (RL) scattering amplitudes are prescribed (see below).

The massless particles \( R_\sigma \) and \( L_\sigma \) are the only stable excitations in SSM\(^{(\pi)}\). Their factorized scattering is defined by the following commutation relations:

\[ R_{\sigma_1}(\beta_1)R_{\sigma_2}(\beta_2) = U_{\sigma_1\sigma_2}^{\sigma_1\sigma_2}(\beta_1 - \beta_2)R_{\sigma_2}(\beta_2)R_{\sigma_1}(\beta_1), \]
\[ L_{\sigma_1}(\beta_1)L_{\sigma_2}(\beta_2) = U_{\sigma_1\sigma_2}^{\sigma_1\sigma_2}(\beta_1 - \beta_2)L_{\sigma_2}(\beta_2)L_{\sigma_1}(\beta_1), \]
\[ R_{\sigma_1}(\beta_1)L_{\sigma_2}(\beta_2) = V_{\sigma_1\sigma_2}^{\sigma_1\sigma_2}(\beta_1 - \beta_2)L_{\sigma_2}(\beta_2)R_{\sigma_1}(\beta_1), \] (1.22)

which introduce the right–right (RR) and left–left (LL) amplitudes \( U_{\sigma_1\sigma_2}^{\sigma_1\sigma_2}(\beta) \) (describing the scattering of two right-movers or two left-movers) and the RL amplitudes \( V_{\sigma_1\sigma_2}^{\sigma_1\sigma_2}(\beta) \). In the massless FST under consideration the RL ampli-
tudes are formally the same as the RR and LL ones. Explicitly,
\[ V_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}(\beta) = U_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}(\beta) = \frac{U_1(\beta)}{\beta - i\pi} \left[ \beta \delta_{\sigma_1}^{\sigma_1} \delta_{\sigma_2}^{\sigma_2} - i\pi \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_2}^{\sigma_1} \right] , \] (1.23)
where \( U_1(\beta) \) is the partial amplitude in the SU(2) isospin-1 channel
\[ U_1(\beta) = \frac{\Gamma \left( \frac{1}{2} + \beta/2i\pi \right) \Gamma \left( -\beta/2i\pi \right)}{\Gamma \left( \frac{1}{2} - \beta/2i\pi \right) \Gamma \left( \beta/2i\pi \right)} . \] (1.24)

Note that under the parameterization (1.21) the right-moving particle \( R_\sigma(\beta) \) becomes soft \((p \to 0)\) if \( \beta \to -\infty \). Contrary, for the left-moving one \( L_\sigma(\beta) \) we reach the soft region when \( \beta \to \infty \). Therefore the soft RL scattering is described by the amplitude \( V_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}(\beta) \) in the limit \( \beta \to -\infty \), where it becomes trivial. It was suggested in ref. [17] that the scattering theory (1.22) with trivial RL amplitudes is in fact the FST of the limiting IR CFT SU(2).1.

The Bethe ansatz equations, which follow from this massless scattering theory, were analysed in ref. [31]. It has been shown that the effective conformal central charge \( c_{\text{eff}}(t) \) predicted by this FST interpolates continuously between the UV value \( c_{\text{UV}} = 2 \) at \( t \to -\infty \) and \( c_{\text{IR}} = 1 \) at large distances where only soft particles contribute. This is what one expects in the SSM\(^{(\pi)}\) field theory.

A crucial observation which initiates the developments of this paper is that the two scattering theories described are in fact special points of two one-parameter families of consistent FST's, which we call SST\(_{\lambda}^{(\pm)}\) and SST\(_{\lambda}^{(-)}\). They vary continuously with a real parameter \( \lambda \) and at \( \lambda = 0 \) turn to the FST's of SSM\(_0^{(\theta)}\) and SSM\(_0^{(\pi)}\) (taking in this notation the names SST\(_0^{(\pm)}\) and SST\(_0^{(-)}\), respectively). The FST's SST\(_\lambda^{(\pm)}\) are described explicitly in sect. 2. Both are invariant under a global U(1) charge symmetry, the U(1) being enlarged up to SU(2) at \( \lambda = 0 \).

It seems very natural to suppose that SST\(_\lambda^{(\pm)}\) are the FST's of some deformation of the O(3) SM (1.10) which preserves the integrability at \( \theta = 0 \) and \( \theta = \pi \). In sect. 3 a one-parameter family of axially symmetric solutions to the one-loop RG equation (1.7) is presented. The axial symmetry is natural in view of the U(1) invariance of SST\(_\lambda^{(\pm)}\). We parameterize the family by a real variable \( \nu \). Every trajectory is UV-stable and at short distances \( t \to -\infty \) the corresponding metric looks like a long sausage, being almost flat in between two round ends where all the curvature is concentrated. One could imagine this long sausage as two Witten's euclidean black holes [32] with their infinite-cylinder ends stuck against each other. As \( t \to -\infty \) the sausage simply lengthens while the shape of its tips is stable and the maximum curvature there remains bounded, \( R_{\text{max}} \sim 4\nu \).

At \( \nu \ll 1 \) this trajectory defines perturbatively an anisotropic SM which we call the sausage sigma model SSM\(_\nu^{(\theta)}\), where \( \theta \) specifies the topological angle in (1.3). In the limit \( \nu \to 0 \) the sausage becomes more and more thick and at \( \nu = 0 \) we end up with a growing (at \( t \to -\infty \)) sphere of SSM\(_0^{(\theta)}\). At \( \nu \neq 0 \)
and $t \to -\infty$ an observer living somewhere in the middle of the sausage sees a frozen picture of infinite flat cylinder of circumference $l \sim \nu^{-1/2}$. This infinite cylinder (compactified however) is a fixed point of eq. (1.6). We expect that the corresponding CFT (which we call RU(1)$_{t}$) is in fact the UV CFT of SSM$^{\theta}_{\nu}$. In particular $c_{UV} = 2$ for the whole family SSM$^{\theta}_{\nu}$.

We find it plausible that SSM$^{\theta}_{\nu}$ is integrable at $\theta = 0$ and $\theta = \pi$ and that the SST$^{(+)}_{\lambda}$ and SST$^{(-)}_{\lambda}$ are just the corresponding FST's (with somehow related parameters $\lambda$ and $\nu$). To justify this conjecture we apply the Bethe ansatz technique to the FST's SST$^{(\pm)}_{\lambda}$. In sects. 4 and 5 two particular observables are analysed for both families SST$^{(\pm)}_{\lambda}$, the specific vacuum energy $F^{(\pm)}(A)$ in the presence of external constant (space-time independent) U(1) gauge field $A_{\mu}$ and the finite-temperature $T = 1/r$ free energy $E^{(\pm)}(r)$. Both the gauge field $A_{\mu}$ and the finite-temperature result effectively in some IR cutoff of order $A'$ and $T'$, respectively. When $A \to \infty$ or $T \to \infty$ the field theory falls into the UV regime. We shall see that the two apparently different FST's, SST$^{(+)}_{\lambda}$ and SST$^{(-)}_{\lambda}$, remarkably lead to the same short-distance perturbative expansions for $F^{(\pm)}(A)$ and $E^{(\pm)}(r)$, the leading UV corrections (rather non-trivial in the finite-temperature case) coinciding at $\lambda \ll 1$ with the perturbative predictions of the SSM$^{\theta}_{\nu}$ field theory. Studying the vacuum energies $F^{(\pm)}(A)$ at $A \to \infty$ we are even able to separate the non-perturbative instanton contributions. They are again the same for SST$^{(+)}_{\lambda}$ and SST$^{(-)}_{\lambda}$ and the only difference between the analytic structures of $F^{(+)}(A)$ and $F^{(-)}(A)$ near $A = \infty$ is that the odd instanton number contributions enter with opposite signs. This is in agreement with (1.4) at $\theta = 0$ and $\theta = \pi$. Moreover, in the one-instanton contribution to $E^{(\pm)}(A)$ (and also to $E^{(\pm)}(r)$, as is extracted numerically in sect. 5) we find a characteristic irregular logarithm. In the lagrangian formulation this logarithm naturally appears as a manifestation of the one-instanton logarithmic divergency [29–31].

All these successful tests hardly leave any doubt about our SM interpretation of FST's SST$^{(\pm)}_{\lambda}$. The only problem is that the lagrangian definition of the sausage sigma model is restricted to the one-loop approximation. It means that the identification between SSM$^{(0)}_{\nu}$ (SSM$^{(\pi)}_{\nu}$) and SST$^{(+)}_{\lambda}$ (SST$^{(-)}_{\lambda}$) is precisely specified only for $\nu \to 0, \lambda \to 0$. In particular, we are only able to find the leading (one-loop) relation between $\nu$ and $\lambda$. Because of the lack of the exact lagrangian formulation we would rather prefer to consider the scattering theories SST$^{(\pm)}_{\lambda}$ as an unambiguous non-perturbative definition of the sausage sigma model at $\theta = 0$ and $\theta = \pi$.

While leading to almost the same short-distance behavior the two FST's SST$^{(\pm)}_{\lambda}$ are quite different on-mass-shell. Whereas SST$^{(+)}_{\lambda}$ has only massive particles, the spectrum of SST$^{(-)}_{\lambda}$ contains massless excitations and therefore SSM$^{(\pi)}_{\nu}$ is expected to be an interpolating field theory with its IR asymptotics controlled
by some CFT. In this context it is relevant to note that SU(2), i.e. the CFT appearing in the IR limit of SSM_{(τ)}\_0, is in fact a point of a one-parameter family of CFT's with the common value c = 1 of the conformal central charge [33]. All the CFT's in the family are invariant with respect to the right and left U(1) \_1 current algebras. Here we use the notation U(1)\_1 for these CFT's (with a positive continuous parameter \( α \)). When \( α = 1 \) the U(1) current algebra is in fact a subalgebra of the SU(2) current algebra, so that in this notation SU(2)\_1 = U(1)\_1.

The corresponding line of fixed points often appears in two-dimensional statistical mechanics, being responsible for the critical universality in a variety of two-dimensional statistical systems. It seems correct to think of the field theories SSM_{(τ)}\_0 as of a continuous family of interpolating flows running from the UV fixed points RU(1)\_1 to the IR fixed points U(1)\_0, as drawn in fig. 2.

To conclude the section we briefly recall the family U(1)\_0. There is a convenient representation in terms of the free massless scalar field \( φ(x) \) with the action

\[ A_{U(1)} = \frac{1}{4\pi} \int (\partial_\mu φ)^2 d^2 x. \] (1.25)

The field \( φ(x) \) is an angular variable (the compactified free field) with the identification

\[ φ(x) \sim φ(x) + \frac{2π}{α}. \] (1.26)

The right and left U(1) current algebras are generated by the currents

\[ I = i\partial φ, \quad \bar{I} = -i\bar{∂}φ. \] (1.27)

Define the following holomorphic–antiholomorphic decomposition of field \( φ \):

\[ φ = ϕ + \bar{ϕ}, \] (1.28)

and its dual

\[ χ = ϕ - \bar{ϕ}, \] (1.29)
which is again a free massless field with the compactification
\[ \chi(x) \sim \chi(x) + 4\pi \alpha. \] (1.30)

Eqs. (1.26) and (1.30) imply the following maximal set of mutually local primary fields:
\[ V_{mn} = \exp \left( i \alpha n \varphi + \frac{im}{2\alpha} \chi \right), \quad m, n \in \mathbb{Z} \] (1.31)
of conformal dimensions
\[ (A_{mn}, \bar{A}_{mn}) = \left( \left( \frac{\alpha n}{2} + \frac{m}{2\alpha} \right)^2, \left( \frac{\alpha n}{2} - \frac{m}{2\alpha} \right)^2 \right). \] (1.32)

If \( \alpha < 1 \) the fields \( \exp(\pm 2i\alpha \varphi) \) (conformal dimension \( A_{\text{IR}} = \alpha^{-2} \)) are relevant. It means that one can observe the CFT \( U(1)_\alpha \) as the UV limit of the sine–Gordon model
\[ \mathcal{A}_{\text{SG}} = \frac{1}{4\pi} \int \left[ (\partial_\mu \varphi)^2 + \frac{m_0^2}{2\alpha^2} \cos 2\alpha \varphi \right] d^2x \] (1.33)
The scalar operators \( \exp(\pm 2i\chi/\alpha) \) of dimension \( A_{\text{IR}} = \alpha^{-2} \) are irrelevant at \( \alpha < 1 \). We shall see that the combination \( \cos(2\chi/\alpha) \) is exactly the field which attracts the trajectory \( \text{SSM}^{(\pi)} \) at its IR fixed point \( U(1)_\alpha \). Therefore the following irrelevant perturbation of \( U(1)_\alpha \)
\[ \mathcal{A}_{\text{IR}} = \frac{1}{4\pi} \int \left[ (\partial_\mu \chi)^2 + \frac{M_0^2\alpha^2}{2} \cos \frac{2}{\alpha} \chi + \text{higher-dimension counterterms} \right] d^2x \] (1.34)
is an effective large-distance action of \( \text{SSM}^{(\pi)} \). This “irrelevant sine–Gordon” generates the leading IR corrections (in powers of the dimensional coupling constant \( M_0 \sim [\text{mass}]^{1-A_{\text{IR}}} \)) to the limiting CFT \( U(1)_\alpha \).

At \( \alpha = 1 \) the collection (1.31) includes the holomorphic and antiholomorphic fields \( \exp(\pm 2i\phi) \) and \( \exp(\pm 2i\bar{\phi}) \). Together with the \( U(1) \) currents (1.27) they form the right and left \( SU(2) \) current algebras
\[ J_0 = I, \quad \bar{J}_0 = \bar{I} \]
\[ J_\pm = e^{\pm 2i\phi}, \quad \bar{J}_\pm = e^{\pm 2i\bar{\phi}} \] (1.35)
of level \( k = 1 \). When studying the action (1.34) at \( \alpha = 1 \) (or in the region \( 1 - \alpha \ll 1 \)) where the perturbation is marginal (or nearly marginal) it is convenient to substitute the sine–Gordon representation (1.34) by the following anisotropic \( J \bar{J} \) perturbation of \( SU(2)_1 \):
\[ \mathcal{A}_{\text{IR}} = \mathcal{A}_{\text{SU(2)_1}} + \frac{f_\parallel}{2\pi} \int J_0 \bar{J}_0 d^2x + \frac{f_\perp}{4\pi} \int (J_+ \bar{J}_- + J_- \bar{J}_+) d^2x. \] (1.36)
Here \( f_\parallel \) and \( f_\perp \) are (small) running coupling constants. The corresponding one-loop RG equations read
\[ \frac{df_\parallel}{dt} = f_\parallel^2, \quad \frac{df_\perp}{dt} = f_\parallel f_\perp. \] (1.37)
In fig. 3 the pattern of trajectories is pictured. It consists of hyperbolas

$$\epsilon^2 = f_{\parallel}^2 - f_{\perp}^2$$  \hspace{1cm} (1.38)

parameterized by the variable $\epsilon$, which can be either real or purely imaginary. The line $f_{\perp} = 0$ is a line of fixed points. The irrelevant trajectories we are interested in are located in the sector $f_{\parallel} \leq 0, |f_{\perp}| \leq f_{\parallel}$. Here the (one-loop) flow of eq. (1.37) has the form

$$f_{\parallel} = -\epsilon \coth \epsilon t,$$
$$f_{\perp} = \pm \frac{\epsilon}{\sinh \epsilon t}$$  \hspace{1cm} (1.39)

with real $\epsilon$. Choosing it to be positive in this sector we find the following (one-loop) relation between the flow parameters $\epsilon$ and $t$ and the sine–Gordon (1.34) couplings $\alpha$ and $M_0$:

$$\epsilon / 2 = 1 - \alpha^{-2} + O(\epsilon^2),$$
$$t = -\frac{1}{\epsilon} \log \frac{M_0^2}{8\epsilon} + O(1).$$  \hspace{1cm} (1.40)

The trajectories (1.39) are used in sects. 4 and 5 in comparison with the Bethe ansatz predictions of massless SST$^{(-)}$.

2. Sausage scattering theories SST$^{(\pm)}$

In this section we formulate the two families of $U(1)$-symmetric FST's SST$^{(\pm)}$. Let us begin with the massive one SST$^{(+)}$. Originally it was discovered in ref. [34] as a $U(1)$-symmetric solution to the factorization (Yang–Baxter) equations of dimension 3. The spectrum consists of three massive particles $A_3$;
The scattering is invariant under the U(1) rotations

\[ A_0 \rightarrow A_0, \quad A_\pm \rightarrow e^{\pm iQ^{(+)}} A_\pm, \]

i.e. the total charge \( \sum_i Q^{(+)} s_i \) is conserved. Here \( 0 \leq \xi < 2\pi \) is the U(1) rotation angle. For later convenience we have introduced the U(1) charges \( \pm Q^{(+) \prime} \) of the particles \( A_\pm \). There is also a charge conjugation symmetry which acts as \( \bar{A}_\pm = A_{\bar{s}} \), where \( \bar{s} = -s \). The two-particle scattering amplitudes \( S_{i,j}^{s_i,s_j}(\beta) \) are defined by the commutation relations

\[ A_{i_1}(\beta_1) A_{j_2}(\beta_2) = S_{i,j}^{s_i,s_j}(\beta_1 - \beta_2) A_{i_2}(\beta_2) A_{j_1}(\beta_1). \]

They are \( C, P \) and \( T \) symmetric:

\[ S_{i,j}^{s_i,s_j}(\beta) = S_{j,i}^{s_j,s_i}(\beta) = S_{i,j}^{s_i,s_j}(\beta) = S_{j,i}^{s_j,s_i}(\beta). \]

and satisfy the crossing symmetry relations

\[ S_{i,j}^{s_i,s_j}(\beta) = S_{j,i}^{s_j,s_i}(i\pi - \beta). \]

With these symmetries we have explicitly

\[ S_{++}^{00}(\beta) = S_{++}^{00}(i\pi - \beta) = \frac{\sinh \lambda(\beta - i\pi)}{\sinh \lambda(\beta + i\pi)}, \]

\[ S_{+0}^{00}(\beta) = S_{+0}^{00}(i\pi - \beta) = -i \frac{\sin 2\pi \lambda}{\sinh \lambda(\beta - 2i\pi)} S_{++}^{00}(\beta), \]

\[ S_{+0}^{+0}(\beta) = -i \frac{\sin \pi \lambda \sin 2\pi \lambda}{\sinh \lambda(\beta - 2i\pi)} S_{++}^{+0}(\beta), \]

\[ S_{00}^{+0}(\beta) = S_{+0}^{00}(\beta) + S_{-+}^{+0}(\beta), \]

where \( \lambda \) is just the variable parameterizing the SST\( ^{(+) \prime} \) family. For every \( \lambda \) these amplitudes are simple trigonometric functions of \( \beta \) (of period \( i\pi /\lambda \)) while their collection (2.5) is easily verified to satisfy the factorization equations. At \( \lambda \rightarrow 0 \) the period is infinite, the amplitudes (2.5) are rational functions of \( \beta \) and in the basis \( A_1 = (A_+ + A_-)/\sqrt{2}; A_2 = -i(A_+ - A_-)/\sqrt{2}; A_3 = A_0 \) do coincide with the O(3)-symmetric ones (1.18)–(1.20).

If \( 0 \leq \lambda \leq 1/2 \) all the amplitudes \( S_{i,j}^{s_i,s_j}(\beta) \) are free of poles inside the physical strip \( 0 \leq \text{Im}\, m\beta < \pi \). In this region of \( \lambda \) the three particles \( A_0, A_\pm \) are only stable excitations and the amplitudes (2.5) completely define the SST\( ^{(+) \prime} \) scattering theory. At \( \lambda = 1/2 \) the scattering (2.5) becomes trivial \( S_{++}^{+0} = S_{++}^{0+} = 1; S_{+0}^{+0} = S_{00}^{00} = 1 \). We have a simple free field theory at this point, which includes a complex massive free fermion \( A_\pm \) and a free boson \( A_0 \) of the same mass. The FST (2.5) becomes more complicated if \( \lambda > 1/2 \). There are some bound-state
poles in the amplitudes and one expects more stable particles in the spectrum. In this paper we restrict ourselves to the “repulsive” region $0 \leq \lambda \leq 1/2$.

Let us turn to the massless $\text{SST}_{\lambda}^{(-)}$. We suppose the same spectrum as in the $\text{SU}(2)$-symmetric case of $\text{SST}_{0}^{(-)}$ with the same energy–momentum parameterization (1.21). There are right- and left-moving massless doublets $R_{\sigma}(\beta)$ and $L_{\sigma}(\beta) \ (\sigma = \pm)$. In general there is no $\text{SU}(2)$-symmetry in the scattering, which respects only $\text{U}(1)$ of simultaneous rotations

$$R_{\pm} \to e^{\pm iQ(-1)^{\lambda}} R_{\pm}, \quad L_{\pm} \to e^{\pm iQ(-1)^{\lambda}} L_{\pm},$$

(2.6)

where $Q^{(-)}$ is the $\text{U}(1)$ charge of massless particles. The FST is again defined by the two-particle commutation relations (1.22). The RR and LL amplitudes $U^{\sigma_{1}\sigma_{2}}_{\pm}(\beta)$, which enjoy the $\text{U}(1)$ charge symmetry and solve the factorization equations in the corresponding sectors, read explicitly

$$U^{++}_{\pm}(\beta) = U^{--}(\beta) = U_{0}(\beta),$$

$$U^{+-}_{\pm}(\beta) = U^{+-}_{\pm}(\beta) = -\frac{\sinh[\lambda \beta/(1 - \lambda)]}{\sinh[\lambda(\beta - i\pi)/(1 - \lambda)]} U_{0}(\beta),$$

$$U^{--}_{\pm}(\beta) = U^{+\pm}_{\pm}(\beta) = -i \frac{\sin[\pi \lambda/(1 - \lambda)]}{\sinh[\lambda(\beta - i\pi)/(1 - \lambda)]} U_{0}(\beta),$$

(2.7)

where

$$U_{0}(\beta) = -\exp i \int_{0}^{\infty} \frac{\sinh[(1 - 2\lambda)\pi \omega/2\lambda] \sin \omega \beta}{\cosh[\pi \omega/2] \sinh[(1 - \lambda)\pi \omega/2\lambda]} \frac{d\omega}{\omega}.$$  

(2.8)

Note that these $U^{\sigma_{1}\sigma_{2}}_{\pm}(\beta)$ coincide formally with the soliton scattering amplitudes in the sine–Gordon model (1.33) at $\alpha^{2} = 1 - \lambda$ [16]. The mixed RL factorization equations together with the analytic crossing-unitarity relations in the RL scattering are satisfied if $V^{\sigma_{1}\sigma_{2}}_{\sigma_{1}\sigma_{2}}(\beta) = U^{\sigma_{1}\sigma_{2}}_{\sigma_{1}\sigma_{2}}(\beta - \beta_{0})$ with an arbitrary shift $\beta_{0}$. For $\text{SST}_{\lambda}^{(-)}$ we take this shift real. Therefore it can be set to zero by a proper normalization of the scale $M$ in eq. (1.21) and we have

$$V^{\sigma_{1}\sigma_{2}}_{\sigma_{1}\sigma_{2}}(\beta) = U^{\sigma_{1}\sigma_{2}}_{\sigma_{1}\sigma_{2}}(\beta).$$  

(2.9)

With this choice the physical strip $0 \leq \text{Im\ } m\beta \leq \pi$ is free of poles if $0 \leq \lambda \leq 1/2$. For $\lambda > 1/2$ poles appear in the physical strip. While one could think of a possible interpretation in terms of RR and LL scattering, they look absolutely intolerable in the RL amplitudes, leading to poles in the physical sheet of the two-particle $s$-channels invariant $s = M^{2} e^{\delta}$. For this reason we restrict our definition (1.22), (2.7)–(2.9) of $\text{SST}_{\lambda}^{(-)}$ to the interval $0 \leq \lambda \leq 1/2$.

At $\lambda = 0$ the $\text{SU}(2)$ symmetry arises and we are back to the FST of $\text{SSM}_{0}^{(\pi)}$ described in sect. 1. An interesting phenomenon occurs at $\lambda = 1/2$. The scattering theory (2.7), (2.8) becomes trivial at this point $U^{++}_{\pm} = U^{+-}_{\pm} = -1;$
$U_{-+}^+ = 0$. At first sight it looks like we deal with a theory of free massless (charged) fermions, i.e. with a CFT of central charge $c = 1$. This is not true however. Consider $\text{SST}^{(-)}_\lambda$ at $1/2 - \lambda << 1$ and notice a pole at $\beta \approx -2\pi (1 - 2\lambda)$ in the RL scattering. It corresponds to a narrow neutral resonance $B_0$ of complex mass $M_{B_0} = M(1 - i\pi(1 - 2\lambda))$ unstable under decay $B_0 \rightarrow R_+ L_-$ or $B_0 \rightarrow R_- L_+$. As $\lambda \rightarrow 1/2$ the pole approaches the s-channel cut while the decay rate of $B_0$ vanishes. Finally at $\lambda = 1/2$ the $B_0$ is a stable neutral boson of mass $M_{B_0} = M$. The limiting $\text{SST}^{(-)}_{1/2}$ is a free field theory of complex massless fermions $R_\pm$ and $L_\pm$ plus massive boson $B_0$. The UV central charge is $c_{UV} = 2$ as for general $\lambda$ (see below).

In the soft limit of RL scattering $\beta \rightarrow -\infty$ the reflection amplitude $V_{-+}^+$ vanishes while $V_{++}^+$ and $V_{+-}^+$ are rapidity independent phases

$$V_{++}^+ = V_{-+}^- = \exp\frac{i\pi}{2(1 - \lambda)},$$
$$V_{+-}^+ = V_{-+}^- = \exp\frac{-i\pi}{2(1 - \lambda)},$$
$$V_{++}^- = V_{+-}^+ = 0. \quad (2.10)$$

Now, let us define a massless FST where RR and LL scattering is described by amplitudes (2.7), (2.8) and RL amplitudes are constant phases (2.10). We believe that this scale-invariant scattering theory provides an FST description of CFT $U(1)_\alpha$ (with $\alpha^2 = 1 - \lambda$). Note that after the duality transformation $R_\sigma \rightarrow R_\sigma; L_\sigma \rightarrow L_{-\sigma}$, this scale-invariant FST coincides with the UV limit of the soliton $S$-matrix in the sine-Gordon model (1.33). The duality transformation is related to the duality between fields $\phi$ and $\chi$ in the actions (1.33) and (1.34).

### 3. Sausage trajectories

Here we present a one-parameter family of explicit axially symmetric solutions to the one-loop RG equation (1.7) (recall that the target space $\mathcal{M}$ has $S^2$ topology). The corresponding metric flows $G_{ij}(t)$ are used in the action (1.3) to define perturbatively (at the one-loop level) the family of $\text{SSM}^{(\sigma)}_\nu$.

In conformal coordinates $X$, $Y$ on $\mathcal{M}$ the metric $G_{ij}(t)$ has the form (1.5). Let $X$ be the angular coordinate, $0 \leq X < 2\pi$, on our surface so that the axially symmetric $\Phi(Y)$ is independent of $X$ and eq. (1.7) reads

$$\frac{\partial \Phi}{\partial t} = \frac{1}{4\pi} e^{-\Phi} \frac{\partial^2 \Phi}{\partial Y^2}. \quad (3.1)$$

Conformal parameterization of the sphere implies that $Y$ covers the whole real axis $-\infty < Y < \infty$, the points $Y = \pm \infty$ corresponding to the "north" and "south" poles of the manifold. The metric is smooth at the poles if

$$\Phi(Y) \sim -2|Y| \quad \text{for} \quad Y \rightarrow \pm \infty \quad (3.2)$$
Fig. 4. Trajectories of eq. (3.5) corresponding to non-singular SM metrics. The sausage trajectories are in the sector $a > b$.

In these coordinates the corresponding axially symmetric action is

$$A_0 = \frac{1}{2} \int e^{\Phi(Y)} \left( (\partial_\mu Y)^2 + (\partial_\mu X)^2 \right) d^2 x + i\theta T.$$  (3.3)

The following simple ansatz:

$$\Phi(Y) = \log \frac{a(t) + b(t) \cosh 2Y}{2}$$  (3.4)

(real and non-singular if $b \geq 0$ and $a \geq -b$) satisfies the boundary condition (3.2) and solves eq. (3.1) if

$$\frac{da}{dt} = \frac{1}{2\pi} b^2,$$

$$\frac{db}{dt} = \frac{1}{2\pi} ab.$$  (3.5)

This system is essentially the same as the "sine–Gordon" system (1.37). We have the same picture of hyperbolic trajectories (fig. 4)

$$\nu^2 = a^2 - b^2.$$  (3.6)

This time we are interested in the UV stable trajectories with $\nu^2 \geq 0$, which begin at the line of UV fixed points $b = 0$, $a > 0$ and fill the sector $b > 0$, $a > b$. In this region one has explicitly

$$a(t) = -\nu \coth \frac{\nu (t - t_0)}{2\pi},$$

$$b(t) = -\nu / \sinh \frac{\nu (t - t_0)}{2\pi}.$$  (3.7)

A few examples of the metric evolution are pictured in fig. 5. For the sake of visualization we embed the surface in three-dimensional euclidean space. While
at $\nu(t_0 - t) \ll 1$ it appears like a shrinking sphere, in the UV limit we see a long expanding sausage of length

$$L \simeq \frac{\sqrt{2\nu}}{2\pi} (t_0 - t).$$

(3.8)

In the middle it tends to a flat cylinder of circumference

$$l = 2\pi \sqrt{2\nu},$$

(3.9)

so that one could identify the fixed point $a = \nu, b = 0$ in fig. 4 with the CFT $\text{RU}(1)$. At $\nu = 0$ we are back to the sphere of the $O(3)$ SM.

Corresponding to every trajectory (3.7) one can write down the following one-loop renormalized action of $\text{SSM}^{(\nu)}$:

$$A_{\text{SSM}^{(\nu)}} = \int \frac{(\partial_{\mu} Y)^2 + (\partial_{\mu} X)^2}{a(t) + b(t) \cosh 2t} \, d^2 x + i\theta T.$$

(3.10)

This one-loop action becomes exact in the limit $\nu \rightarrow 0$, $t \rightarrow -\infty$ with $\nu t$ fixed (we call it the scaling limit). Here (3.10) gives exact asymptotics which could be compared with that predicted by the scattering theories of sect. 2.

It is worth mentioning that the sausage action (3.10) admits also a simple parameterization in terms of the unit-vector field $n_a(x)$ introduced in eq. (1.10) for the $O(3)$ case,

$$A_{\text{SSM}^{(\nu)}} = \frac{1}{2g(t)} \sum_{a=1}^3 \int \frac{(\partial_{\mu} n_a)^2 \, d^2 x}{1 - \nu^2 n_a^2/2g^2(t)} + i\theta T,$$

(3.11)
where now
\[ g(t) = \frac{\nu}{2} \coth \frac{\nu(t_0 - t)}{4\pi}. \quad (3.12) \]

4. Sausage model in magnetic field

In this and the subsequent sections some calculations are performed in order to support the identification of \( \text{SST}_{\vec{z}}^{(+)} \) and \( \text{SST}_{\vec{z}}^{(-)} \) as the FST's of the sausage sigma model at \( \theta = 0 \) and \( \theta = \pi \). First we study the infinite volume specific vacuum energy \( F(A) \) in a constant external magnetic (gauge) field \( A \). The short-distance (perturbative) behavior for \( A \to \infty \) is mostly under attention.

The action (3.3) is symmetric with respect to global axial rotations of the sausage
\[ X(x) \to X(x) + \xi, \quad 0 \leq \xi < 2\pi. \quad (4.1) \]
It is possible therefore to couple \( \text{SSM}_\nu^{(\theta)} \) to external abelian gauge field \( A_\mu \). The corresponding euclidean (one-loop) action is
\[ A_{\text{SSM}_\nu^{(\theta)}}(A) = \int \frac{(\partial_\mu Y)^2 + (\partial_\mu X - iA_\mu)^2}{a + b \cosh 2Y} d^2x + i\theta T. \quad (4.2) \]

Hence forward we have in mind that \( A_\mu \) is space-time independent and analyse the corresponding bulk free energy \( F^{(\theta)}(A) \). It is readily seen in perturbation theory that the external field acts as an IR cutoff scale \( A = |A_\mu| \). In eq. (3.7) we therefore assign
\[ t_0 - t = \log \frac{A}{A} \quad (4.3) \]
where \( A \) is some normalization parameter which is expected to be of order of the mass scale. To ensure the validity of our one-loop action (4.2) we go to the scaling limit \( \nu \to 0, A/A \to \infty \) with the scaling variable
\[ z = \frac{\nu}{2\pi} \log \frac{A}{A} \quad (4.4) \]
fixed. The topological term has no influence in this limit and we get from (4.2)
\[ F^{(\theta)}(A) = -\frac{A^2}{a + b} = -\frac{A^2}{\nu} \tanh \frac{z}{2}. \quad (4.5) \]
The next (two-loop) correction is of relative order \( \sim \nu \log \nu \). Its development remains to be performed and requires the two-loop corrected sausage metric which would solve the two-loop RG equation (1.6). In the extreme UV limit \( z \to \infty \) we have from (4.5)
\[ \frac{F^{(\theta)}(A)}{A^2} = -\frac{1}{\nu} \left( 1 - 2 \left( \frac{A}{A} \right)^{\nu/2\pi} + 2 \left( \frac{A}{A} \right)^{2\nu/2\pi} - \ldots \right). \quad (4.6) \]
The \( \theta \)-dependence of \( F^{(\theta)}(A) \) shows up in the non-perturbative instanton contributions. Up to the one-loop renormalization these can be estimated along
the lines of refs. [29,30]. The general $q$-instanton (plus $q$-anti-instanton) term behaves as ($|q| \neq 1$)

$$\frac{F_q(\theta)(A)}{A^2} = -2 \cos \theta \, e^{2|q|\nu/2}\nu/2\pi \left[ I_q(\nu) + \text{higher-loop corrections} \right], \quad (4.7)$$

where $I_q(\nu)$ are some (inaccessible at the one-loop level) $A$-independent constants (the higher loop corrections bear some perturbative $A$-dependence $\sim (A/A)^{\nu/2\pi}$). Note that contrary to the “regular” UV expansion (4.6) in powers of $(A/A)^{\nu/2\pi}$ the instantons produce irregular terms like $\sim (A/A)^{2|q|}$ in the UV analytic structure of $F(\theta)(A)/A^2$. One exceptional case is the one-instanton contribution which has a logarithmic divergence in the integration over small-instanton configurations [29–31]. This leads to the characteristic logarithm in the one-instanton term

$$\frac{F_1(\theta)(A)}{A^2} = 2 \cos \theta \left( \frac{A}{A} \right)^2 \left[ I_{(\log)} \log A r_0 + I_1 + O \left( (A/A)^{\nu/2\pi} \right) \right], \quad (4.8)$$

where $r_0$ is the UV cutoff scale.

Corresponding to the axial symmetry of (4.2) there is a conserved abelian current

$$j_\mu = e^{\Phi(y)}(\partial_\mu X - i A_\mu). \quad (4.9)$$

Let us direct $A_\mu$ along the “time” coordinate $x_2$. In the hamiltonian treatment this external field is coupled to the conserved charge

$$Q = \int_{-\infty}^{\infty} j_0 \, dx_1, \quad (4.10)$$

i.e.

$$\mathcal{H}_A = \mathcal{H}_0 - AQ. \quad (4.11)$$

At this point we are ready to turn to our FST’s SST$^{(\pm)}_{\lambda}$. Now the conserved charge $Q$ is just the U(1) charge of the particles $A_\pm$ in SST$^{(\pm)}_{\lambda}$ or $R_\sigma, L_\sigma$ in SST$^{(\pm)}_{\lambda}$. In general we see no reason to identify the $2\pi$-rotation of the sausage with that of particles $A_\pm$ in SST$^{(\pm)}_{\lambda}$. Therefore in (2.1) the corresponding charge

$$Q(\pm)(\lambda)$$

is introduced,

$$QA_\pm = s Q(\pm)(\lambda) A_\pm, \quad (4.12)$$

which may depend on $\lambda$. At $\lambda = 0$ the charge is prescribed by the O(3) symmetry

$$Q(\pm)(0) = 1. \quad (4.13)$$

Analogously in SST$^{(-)}_{\lambda}$

$$QR_\sigma = \sigma Q^{(-)}(\lambda) R_\sigma, \quad QL_\sigma = \sigma Q^{(-)}(\lambda) L_\sigma. \quad (4.14)$$

Again the SU(2) invariance at $\lambda = 0$ requires

$$Q^{(-)}(0) = 1/2 \quad (4.15)$$
In the further developments we closely follow the Bethe ansatz technique of refs. [20,21]. Begin with the massive $SU^+_{\lambda}$ and denote

$$h = AQ^{(+)}.$$  

(4.16)

If $h > m$ a sea of particles $A_+ (\beta)$ is excited in the ground state which fill all the possible volume specific ground state energy

$$\mathcal{F}^{(+)}(h) = F^{(+)}(A) - F^{(+)}(0) = -\frac{m}{2\pi} \int_{-B}^{B} \cosh \beta \epsilon (\beta) \, d\beta$$  

(4.17)

amounts to solve (inside the Fermi interval) the following Bethe ansatz integral equation

$$h - m \cosh \beta = \int_{-B}^{B} \tilde{K}(\beta - \beta') \epsilon (\beta') \, d\beta', \quad -B < \beta < B$$  

(4.18)

for a positive function $\epsilon (\beta) > 0$ defined at $-B < \beta < B$. Thus the Fermi boundary $B$ is to be determined from the condition

$$\epsilon (\pm B) = 0.$$  

(4.19)

Kernel $\tilde{K}(\beta)$ is related to the $A_+ A_+$ scattering phase (2.5)

$$\tilde{K}(\beta) = \delta (\beta) - \frac{1}{2i\pi} \frac{d}{d\beta} \log S^{++}_{++} (\beta)$$

$$= \delta (\beta) - \frac{\lambda}{\pi \cosh 2\beta \beta - \cos 2\pi \lambda},$$  

(4.20)

and has the following Fourier transform:

$$K(\omega) = \int \tilde{K}(\beta) e^{i\omega\beta} \, d\beta = \frac{2 \sinh [\pi \omega/2] \cosh [\gamma \omega/2]}{\sinh [(\pi + \gamma) \omega/2]},$$  

(4.21)

where we have introduced new parameter $\pi < \gamma < \infty$

$$\frac{\gamma}{\pi} = \frac{1}{\lambda} - 1.$$  

(4.22)

The leading $h \to \infty$ behavior of $\mathcal{F}^{(+)}(h)$ is readily estimated by the following simple trick. First notice that in this limit $B \to \infty$ while the main contribution to

$$\mathcal{F}^{(+)}(h) = -\frac{m}{2\pi} \int_{-B}^{B} \epsilon (\beta) \, d\beta$$  

(4.23)

comes from the region $B - \beta \approx 1$ near the right boundary. Considering this region for $B \gg 1$ one can safely forget about the left boundary at $\beta = -B$ and
replace eq. (4.18) by

\[ h - \frac{m}{2} e^\beta = \int_{-\infty}^{B} \tilde{K}(\beta - \beta') \epsilon(\beta') \, d\beta'. \] (4.24)

Differentiating this with respect to \( \beta \) and taking (4.19) into account one finds also \( (\epsilon'(\beta) = \frac{d}{d\beta}(\beta) d\beta') \)

\[ -\frac{m}{2} e^\beta = \int_{-\infty}^{B} \tilde{K}(\beta - \beta') \epsilon'(\beta') \, d\beta'. \] (4.25)

Then one substitutes this equation into (4.23)

\[ F(\beta)(h) = \frac{1}{\pi} \int_{-\infty}^{B} \int_{-\infty}^{B} \tilde{K}(\beta - \beta') \epsilon'(\beta') \epsilon(\beta) \, d\beta \, d\beta'. \]

\[ = \frac{h}{\pi} \int_{-\infty}^{B} \epsilon'(\beta) \, d\beta - F(\beta)(h). \] (4.26)

Therefore

\[ F(\beta)(h) = \frac{h}{2\pi} \int_{-\infty}^{B} \epsilon'(\beta) \, d\beta = -\frac{h \epsilon(-\infty)}{2\pi}. \] (4.27)

From eq. (4.24) it is clear that \( \epsilon(-\infty) = h/K(0) \) so that we have

\[ F(\beta)(h) = -\frac{h^2}{2\pi K(0)} = -\frac{(AQ(\beta))^2}{4\pi \lambda}. \] (4.28)

Comparing this behavior with the leading term in (4.5) we find the one-loop relation between the parameters \( \nu \) and \( \lambda \) (recall that \( Q(\beta) = 1 + O(\lambda) \))

\[ \nu = 4\pi \lambda + O(\lambda^2). \] (4.29)

More detailed information about \( F(\beta)(h) \) is obtained by the Wiener–Hopf (WH) technique [35] which permits a systematic large-\( h \) expansion. This technique is applied to eq. (4.18) thanks to the factorizability of the kernel (4.21)

\[ K(\omega) = \frac{1}{K_+(\omega) K_-^{(\omega)}}. \] (4.30)

where

\[ K_+(\omega) = \sqrt{\frac{1}{2(\pi + \gamma)}} e^{i\omega \beta} \frac{\Gamma\left(\frac{i\omega}{2}\right) \Gamma\left(\frac{1}{2} + \frac{i\gamma \omega}{2\pi}\right)}{\Gamma\left(\frac{i(\pi + \gamma) \omega}{2\pi}\right)} \] (4.31)

has neither zeroes nor poles in the lower half-plane \( \text{Im} \omega < 0 \) while

\[ K_-^{(\omega)} = K_+(\omega) \] (4.32)
does the same in the upper half-plane. In (4.31)
\[ \Delta = \frac{\pi + \gamma}{2\pi} \log \frac{\pi + \gamma}{\pi} - \frac{\gamma}{2\pi} \log \frac{\gamma}{\pi} \] (4.33)
so that
\[ K_+ (\omega) = 1 + O(1/\omega) \] (4.34)
as \( \omega \to \infty \) everywhere except for the positive imaginary axis, where all zeroes
and poles are located.

In the WH approach (4.18) is reduced to the following linear integral equation
for a meromorphic function \( v(\omega) \):
\[ v(k) = -i h K_+(0) + \frac{ime^B}{2} K_+(-i) + \int_{C_+} \frac{e^{2i\omega B}}{k + \omega} \alpha(\omega) v(\omega) \frac{d\omega}{2\pi i}, \] (4.35)
while the boundary condition (4.19) reads
\[ ih K_+(0) - \frac{ime^B}{2} K_+(-i) = \int_{C_+} \frac{e^{2i\omega B}}{\omega - i} \alpha(\omega) v(\omega) \frac{d\omega}{2\pi i}. \] (4.36)

Here
\[ \alpha(\omega) = \frac{K_+(\omega)}{K_-(\omega)} = e^{2i\omega A} \frac{\Gamma \left( \frac{i\omega}{2} \right) \Gamma \left( \frac{1}{2} + \frac{i\gamma\omega}{2\pi} \right) \Gamma \left( \frac{1}{2} - \frac{i\gamma\omega}{2\pi} \right) \Gamma \left( \frac{(\pi + \gamma)\omega}{2\pi} \right)}{\Gamma \left( \frac{-i\omega}{2} \right) \Gamma \left( \frac{1}{2} - \frac{i\gamma\omega}{2\pi} \right) \Gamma \left( \frac{1}{2} + \frac{i\gamma\omega}{2\pi} \right) \Gamma \left( \frac{(\pi + \gamma)\omega}{2\pi} \right)} \] (4.37)
and the integration contour \( C_+ \) encircles all the singularities on the positive
imaginary axis whereas the pole at \( \omega = -k \) is outside. Finally the vacuum
energy (4.17) is
\[ \mathcal{J}^{(+)}(\hbar) = -\frac{me^B}{2\pi} K_+(-i) \times \left[ h K_+(0) - \frac{me^B}{4} K_+(-i) + \int_{C_+} \frac{e^{2i\omega B}}{\omega - i} \alpha(\omega) v(\omega) \frac{d\omega}{2\pi i} \right], \] (4.38)
where the explicit pole at \( \omega = i \) is also inside the integration contour \( C_+ \).

In the UV region \( \hbar \gg m \) the boundary \( B \sim \log \hbar/m \) is large and one can treat
the integral terms in eqs. (4.35), (4.36) as small corrections subject to iterative
procedure. At zero order we get again (4.28). For the next iterations we pick the
singularities inside the integration contours which are
(1) Poles of \( \alpha(\omega) \) at
\[ \omega = \frac{i\pi}{\gamma} (2n + 1), \quad n = 0, 1, \ldots \] (4.39)
with residues

\[
a_{2n+1} = \frac{i \pi}{2} e^{2\pi \delta(2n+1)/\gamma} \text{res}_{\omega = \pi(2n+1)/\gamma} \alpha(\omega) \frac{(-\pi)^n}{(n!)^2} \frac{\Gamma\left(\frac{\pi + \gamma}{2\gamma}(2n + 1)\right)}{\Gamma\left(\pi(2n + 1)/\gamma\right)} \frac{\Gamma\left(-\frac{\pi + \gamma}{2\gamma}(2n + 1)\right)}{\Gamma\left(-\frac{\pi + \gamma}{2\gamma}\right)}.,
\]

(4.40)

(2) Poles of \( \alpha(\omega) \) at

\[
\omega = 2in, \quad n = 1, 2, \ldots
\]

with

\[
b_{2n} = \frac{i}{2} e^{4\pi \delta} \text{res}_{\omega = 2in} \alpha(\omega) = \frac{(-\pi)^n}{n!(n-1)!} \frac{\Gamma\left(\frac{1}{2} - \frac{n\gamma}{\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{n\gamma}{\pi}\right)} \frac{\Gamma\left(-\frac{n\gamma}{\pi}\right)}{\Gamma\left(-\frac{n\gamma}{\pi}\right)}.,
\]

(4.42)

(3) A single pole of \( \nu(\omega) \) at \( \omega = i \)

\[
\text{res}_{\omega = i} \nu(\omega) = \frac{im e^{\beta}}{2} K_+(-i).
\]

In the iterative procedure the first series of poles (4.39) produce "perturbative" terms of order \((m/\hbar)^{2n\gamma/\gamma}\), \(n = 1, 2, \ldots\). The leading UV correction comes from the first pole at \( \omega = i\pi/\gamma \)

\[
\mathcal{F}^{(+)}(\hbar) = \frac{\hbar^2}{4\pi \lambda} \left(1 - 4 \left(\frac{1 - \lambda}{1 - 2\lambda}\right)^2 a_1 \left(\frac{\lambda m}{2\hbar}\right)^{2\lambda/(1 - \lambda)} + \ldots\right)
\]

(4.44)

with

\[
a_1 = \frac{\Gamma\left(-\frac{\lambda}{2 - 2\lambda}\right) \Gamma\left(\frac{1}{2 - 2\lambda}\right)}{\Gamma\left(\frac{\lambda}{2 - 2\lambda}\right) \Gamma\left(-\frac{1}{2 - 2\lambda}\right)}.
\]

(4.45)

The exponent here is in agreement with the perturbative expansion (4.6) and relation (4.29). Unfortunately the one-loop approximation of eq. (4.6) does not allow to relate parameters \(A\) and \(m\) even to the leading order in \(\lambda\). This would require the two-loop correction to (4.5). At \(\nu = 0\) (the O(3) sigma model) this correction was found in ref. [21], from where we read off

\[
A = \frac{e^{3/2} m}{8} \left(1 + O(\lambda)\right).
\]

(4.46)

Besides the perturbative terms \(\sim (m/\hbar)^{2n\lambda/(1 - \lambda)}\) the UV expansion of \(\mathcal{F}^{(+)}(\hbar)\) contains "irregular" ones generated by the poles (4.41) and also by the "special" one at \(\omega = i\). They introduce integer powers of \((m/\hbar)^2\) so that in general the weights like \((m/\hbar)^{2q + 2n\lambda/(1 - \lambda)}\) are produced with integer \(q\) and \(n\). Note that the
parity of $q$ depends on the even or odd number of iterations through the special pole $\omega = i$. The whole analytic structure of $F^{(+)}(h)$ near $h/m = \infty$ can be arranged as follows:

$$F^{(+)}(h) = -m^2 \mathcal{E}_0 - \frac{h^2}{\pi} \sum_{q=0}^{\infty} \left( \frac{m}{2h} \right)^{2q} f^{(q)} \left( \frac{h}{m} \right), \quad (4.47)$$

where in general $f^{(q)}(h/m)$ are regular series in $(m/2h)^{2\lambda/(1-\lambda)}$

$$f^{(q)} \left( \frac{h}{m} \right) = \sum_{n=0}^{\infty} f^{(q)}_n \left( \frac{m}{2h} \right)^{2n\lambda/(1-\lambda)} \quad (q \neq 1). \quad (4.48)$$

For example

$$f^{(0)} \left( \frac{h}{m} \right) = \frac{1}{2\lambda} \left( 1 - 4 \left( \frac{1-\lambda}{1-2\lambda} \right)^2 a_1 \left( \frac{\lambda m}{2h} \right)^{2\lambda/(1-\lambda)} \right). \quad (4.49)$$

For the reasons to be explained shortly we have separated in eq. (4.47) some piece $m^2 \mathcal{E}_0$ from the $q = 1$ term. The structure of expansions (4.47), (4.48) conforms to what has been speculated above by the instanton considerations in SSM$_{\nu}^{(0)}$. $f^{(q)}_0$ is naturally interpreted as the leading $q$-instanton contribution while the rest regular terms in (4.48) are the corresponding perturbative corrections. The term $q = 1$ requires slightly more attention since in the integrand of eq. (4.38) there is a double pole at $\omega = i$ which results in a special logarithmic contribution. Evaluating this pole to the leading order we find

$$-\frac{m^2}{16} \cot \frac{\pi}{2\lambda} + \frac{m^2}{4\pi} \left( \log \frac{2h}{m\lambda} + 1 + \frac{1}{2} \psi \left( \frac{1}{2} \right) - \frac{1}{2} \psi \left( \frac{1}{2\lambda} \right) \right) \quad (4.50)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. We would like to isolate the first $\cot \pi/2\lambda$ term of eq. (4.47) as $\mathcal{E}_0$

$$\mathcal{E}_0 = \frac{1}{16} \cot \frac{\pi}{2\lambda}, \quad (4.51)$$

so that the expansion of $f^{(1)}(h/m)$ begins with

$$f^{(1)} \left( \frac{h}{m} \right) = \left( \frac{m}{2h} \right)^2 \times \left( -\log \frac{2h}{m\lambda} - 1 - \frac{1}{2} \psi \left( \frac{1}{2} \right) - \frac{1}{2} \psi \left( \frac{1}{2\lambda} \right) + \sum_{n=1}^{\infty} f^{(1)}_n \left( \frac{m}{2h} \right)^{2n\lambda/(1-\lambda)} \right). \quad (4.52)$$

The logarithm here is readily traced back to the one-instanton logarithmic divergence (4.8). Comparing (4.47) with the perturbative results in SSM$_{\nu}^{(0)}$ we obtain the zero-field bulk vacuum energy $F^{(+)}(0)$ (which has been subtracted in eq. (4.17)) in terms of the physical mass $m$,

$$F^{(+)}(0) = m^2 \mathcal{E}_0 + \mathcal{E}_{\text{inst}}, \quad (4.53)$$
where
\[ \varepsilon_{\text{inst}} = \frac{m^2}{4\pi} \left( \log mr_0 + \text{const} \right). \]

In the scaling limit \( \lambda \to 0 \), \( \log (h/m) \to \infty \) with \( \lambda \log (m/h) \) fixed only the "perturbative" poles (4.39) contribute. Our WH system (4.35), (4.36) is simplified in this limit and admits an exact solution. Corrections to the scaling behavior also can be systematically developed. Details of this calculation will be published elsewhere. Here we quote the scaling limit of \( \mathcal{F}^{(+)}(h) \) together with the leading ("two-loop") correction
\[ \mathcal{F}^{(+)}(h) = -\frac{h^2}{2\pi \lambda} \frac{1-q}{1+q} \left[ 1 + 4\lambda \frac{q}{1-q^2} \log \frac{1-q}{2\lambda} + O(\lambda^2 \log^2 \lambda) \right], \]

where
\[ q = \left( \frac{m \varepsilon^{3/2}}{8h} \right)^{2\lambda/(1-\lambda)}. \]

The scaling function \( (1-q)/(1+q) \) is in agreement with the one-loop perturbative one (4.5) for \( \text{SSM}_{\nu}^{(\beta)} \). At \( \lambda \to 0 \) we recover the result of ref. [21] for \( \text{O}(3) \) SM.

Next we take the massless \( \text{SST}_{\lambda}^{(-)} \) and denote
\[ H = A Q^{(-)}. \]

Now there is no gap in the spectrum and particles \( R_+ (\beta) \) and \( L_+ (\beta) \) are always excited once \( H > 0 \). In the ground state these right- and left-movers fill respectively the semi-infinite rapidity intervals \( -\infty < \beta < B \) and \( -B < \beta < \infty \) with some Fermi boundary \( B \sim \log H/M \). Instead of (4.18) we have the following system of integral equations:
\[
\begin{align*}
H - \frac{Me^\beta}{2} &= \varepsilon_1 (\beta) - \int_{-\infty}^{B} \tilde{k}_1 (\beta - \beta') \varepsilon_1 (\beta') \, d\beta' \\
&\quad - \int_{-B}^{\infty} \tilde{k}_2 (\beta - \beta') \varepsilon_2 (\beta') \, d\beta', \\
&\quad - \int_{-\infty}^{\infty} \tilde{k}_2 (\beta - \beta') \varepsilon_1 (\beta') \, d\beta', \quad -B < \beta < \infty
\end{align*}
\]
\[
\begin{align*}
H - \frac{Me^{-\beta}}{2} &= \varepsilon_2 (\beta) - \int_{-B}^{\infty} \tilde{k}_1 (\beta - \beta') \varepsilon_2 (\beta') \, d\beta' \\
&\quad - \int_{-\infty}^{B} \tilde{k}_2 (\beta - \beta') \varepsilon_1 (\beta') \, d\beta', \\
&\quad - \int_{-\infty}^{\infty} \tilde{k}_2 (\beta - \beta') \varepsilon_2 (\beta') \, d\beta', \quad -B < \beta < \infty
\end{align*}
\]

with the boundary conditions
\[ \varepsilon_1 (B) = \varepsilon_2 (-B) = 0. \]
For the scattering theory (2.7)–(2.9)

\[ \tilde{k}_1(\beta) = \tilde{k}_2(\beta) = \frac{1}{2\pi i} \frac{d}{d\beta} \log U_0(\beta). \]  

(4.60)

In terms of the Fourier transform

\[ k(\omega) = \int \tilde{k}_1(\beta) e^{i\omega \beta} d\beta \]  

(4.61)

the kernel reads explicitly

\[ k(\omega) = \frac{\sinh \frac{1}{2}(\gamma - \pi)\omega}{2 \cosh \frac{1}{2}\pi \omega \sinh \frac{1}{2}\gamma \omega}. \]  

(4.62)

where parameter (4.22) is used again. The vacuum energy \( \mathcal{F}^{(-)}(H) \) is evaluated as

\[ \mathcal{F}^{(-)}(H) = F^{(-)}(A) - F^{(-)}(0) = -\frac{M}{2\pi} \int_{-\infty}^{B} e^\beta \epsilon_1(\beta) d\beta. \]  

(4.63)

Eqs. (4.18) and (4.58) look quite different. One can readily see however that they are very similar in the UV region \( A \rightarrow \infty \). For, at \( H \gg M \) the right and left Fermi intervals strongly overlap at \(-B < \beta < B\). Keeping far enough from say the left Fermi boundary \(-B\) one can neglect its influence and solve (4.58) for \( \epsilon_2 \) by the Fourier transform. This results in the following equation for \( \epsilon_1(\beta) \) near the boundary \( B \)

\[ \frac{H}{1-k(0)} - \frac{M}{2} e^\beta = \int_{-\infty}^{B} \tilde{K}(\beta - \beta') \epsilon_1(\beta') d\beta', \quad \beta < B, \]  

(4.64)

where

\[ K(\omega) = \int \tilde{K}(\beta) e^{i\omega \beta} d\beta = \frac{1 - 2k(\omega)}{1 - k(\omega)} \]  

(4.65)

is precisely the same kernel (4.21) as in eq. (4.18). One expects therefore that the UV structures of \( \mathcal{F}^{(+)}(h) \) and \( \mathcal{F}^{(-)}(H) \) are closely related. In particular at \( H \rightarrow \infty \) we get

\[ \mathcal{F}^{(-)}(H) = -\frac{H^2}{4\pi \lambda (1 - k(0))^2} = -\frac{(2(1 - \lambda) Q^{(-)}(A))^2}{4\pi \lambda}. \]  

(4.66)

Comparing this with eq. (4.28) one recovers exact relation between the “massive” and “massless” charges in SSM\(_v^{(0)}\) and SSM\(_v^{(\infty)}\) in terms of \( \lambda \)

\[ \frac{Q^{(+)}(\lambda)}{Q^{(-)}(\lambda)} = 2(1 - \lambda) \]  

(4.67)

in agreement with (4.13) and (4.15). Note that eqs. (4.28) and (4.66) imply the following exact relation between the charges \( Q^{(\pm)}(\lambda) \) and the limiting UV
circumference of the sausage (i.e. the parameter $l$ of the UV CFT RU $RU(1)_1$):

$$l = \sqrt{\frac{2\pi}{\lambda}} Q^{(+)}(\lambda) = 2(1 - \lambda) \sqrt{\frac{2\pi}{\lambda}} Q^{(-)}(\lambda).$$  \hfill (4.68)

For more details about the UV structure of $\mathcal{F}^{(-)}(H)$ we apply again the WH technique to reduce the system (4.58) to

$$u(k) = -\frac{i h K_+(0)}{k} + \frac{i M e^B K_+(-i)}{2} + \frac{\alpha(\omega)}{k + \omega} \frac{\alpha(\omega)}{2\pi i} u(\omega) d\omega, \hfill (4.69)$$

where we use (consistent with (4.16) and (4.67)) notation

$$h = 2(1 - \lambda) H.$$  \hfill (4.70)

This equation is quite similar to the “massive” one (4.35) with the same $K_+(\omega)$ and the same contour $C_+$. The only difference is in the kernel function which is now

$$\tilde{\alpha}(\omega) = \alpha(\omega) (1 - K(\omega)).$$  \hfill (4.71)

The boundary condition (4.59) acquires the form

$$ihK_+(0) - \frac{i M e^B}{2} K_+(-i) = \int_{C_+} e^{i\omega B} \tilde{\alpha}(\omega) u(\omega) \frac{d\omega}{2\pi i},$$  \hfill (4.72)

and the vacuum energy $\mathcal{F}^{(-)}(H)$ reads

$$\mathcal{F}^{(-)}(H) = -\frac{M e^B}{2\pi} K_+(-i) \times \left[ hK_+(0) - \frac{M e^B}{4} K_+(-i) + \int_{C_+} e^{i\omega B} \tilde{\alpha}(\omega) u(\omega) \frac{d\omega}{2\pi i} \right].$$  \hfill (4.73)

There is no need to reanalyse the $H \gg M$ structure of this WH system. It is sufficient to note that the difference between the kernel functions in (4.35) and (4.69) $K(\omega)\alpha(\omega) = 1/K^2(\omega)$ is free of singularities in the upper half-plane. Therefore the only modification made in eqs. (4.69) and (4.72) is at the special pole $\omega = i$ where

$$\tilde{\alpha}(i) = \alpha(i) (1 - K(i)) = -\alpha(i).$$  \hfill (4.74)

The effect is a minus sign in every iteration through this pole, which results in the alternation of all the “odd instanton number” terms. A slightly more careful inspection of the double pole $\omega = i$ in eq. (4.73) reveals a single term in the leading contribution (4.50) which is not alternated. This is just the one separated in expansion (4.47) so that in the massless case we have

$$\mathcal{F}^{(-)}(H) = -M^2 e_0 - \frac{h^2}{\pi} \sum_{q=0}^{\infty} (-)^q \left( \frac{M}{2h} \right)^{2q} f(q) \left( \frac{h}{M} \right),$$  \hfill (4.75)
where $\epsilon_0$ is again (4.51) and $f^{(q)}(h/M)$ are precisely the same functions as in eq. (4.47). We see that our interpretation of $\text{SST}_{\hat{t}}^{(+)}$ and $\text{SST}_{\hat{t}}^{(-)}$ as SM's at $\theta = 0$ and $\pi$ is quite consistent once $f^{(q)}$ are accepted as the $q$-instanton contributions. In particular this consistence implies the exact equality of the physical scale parameters $m$ and $M$ in $\text{SSM}^{(0)}_\nu$ and $\text{SSM}^{(\pi)}_\nu$. In the massless case the zero-field vacuum energy becomes

$$F^{(-1)}(0) = \frac{M^2}{16} \cot \frac{\pi}{{\tilde{\lambda}}} - \frac{M^2}{4\pi} (\log M r_0 + \text{const.})$$

so that the first term (common for (4.53) and (4.76)) is the even instanton number contribution to the bulk vacuum energy while the second one is due to odd $q$ and carries the one-instanton divergency.

Unlike the massive case where $\mathcal{F}^{(+)}(h) = 0$ for $h \lesssim m$ the massless $\text{SST}_{\hat{t}}^{(-)}$ develops some non-trivial structure in the IR region $H \ll M$. Only soft massless particles are excited in this limit. The Fermi boundary $B \sim \log H/M$ in eqs. (4.58) becomes large negative and one can neglect the interaction between $\epsilon_1(\beta)$ and $\epsilon_2(\beta)$ arriving at two independent scale invariant equations of the form

$$H - \frac{M}{2} e^{\beta} = \int_{-\infty}^{B} \tilde{N}(\beta - \beta') \epsilon_1(\beta') \, d\beta', \quad -\infty < \beta < B$$

(similarly for $\epsilon_2(\beta)$) where

$$\tilde{N}(\beta) = \int N(\omega) e^{i\omega \beta} \, d\beta$$

with

$$N(\omega) = 1 - k(\omega) = \frac{\sinh \frac{1}{2} (\gamma + \pi) \omega}{2 \cosh \frac{1}{2} \omega \pi \sinh \frac{1}{2} \gamma \omega}.$$  

(4.79)

From (4.77) one immediately gets the limiting IR behavior

$$\mathcal{F}^{(-)}(H) = -\frac{H^2}{2\pi N(0)} = -\frac{(1 - \hat{\lambda}) H^2}{\pi},$$

(4.80)

i.e. just what one expects in $\text{U}(1)_\alpha$ with

$$\alpha^2 = 1 - \hat{\lambda}.$$  

(4.81)

The kernel (4.79) admits the factorization

$$N(\omega) = \frac{1}{N_+(\omega) N_-(\omega)},$$

(4.82)

where $N_-(\omega) = N_+(-\omega)$ and $N_+(\omega)$ is analytic in the lower half-plane of $\omega$

$$N_+(\omega) = \sqrt{\frac{2\pi(\gamma + \pi)}{\gamma}} e^{-i\omega \hat{\gamma}} \frac{\Gamma\left(\frac{1}{2} + i\omega\right)}{\Gamma\left(\frac{1}{2} - i\omega\right)} \frac{\Gamma\left(\frac{\pi + \gamma}{2\pi}\omega\right)}{\Gamma\left(\frac{\pi - \gamma}{2\pi}\omega\right)}.$$  

(4.83)
with the same $\lambda$ as in eq. (4.33). The system (4.58) is transformed now to the following WH form convenient for systematic study of the IR corrections at $B \to -\infty$:

$$w(k) = \frac{i H N_+(0)}{k} - \frac{i M e^B N_+(-i)}{k + i} + \int_{C_+} e^{-2i\omega B} \rho(\omega) w(\omega) \frac{d\omega}{2\pi i}$$

$$- i H N_+(0) + \frac{i M e^B}{2} N_+(-i) = \int_{C_+} e^{-2i\omega B} \rho(\omega) w(\omega) \frac{d\omega}{2\pi i},$$

$$\mathcal{F}(-H) = - \frac{M e^B}{2\pi} N_+(-i)$$

$$\times \left[ H N_+(0) - \frac{M e^B}{4} N_+(-i) + \int_{C_+} e^{-2i\omega B} \rho(\omega) w(\omega) \frac{d\omega}{2\pi i} \right],$$

where

$$\rho(\omega) = \frac{N_-(\omega)}{N_+(\omega)} = e^{2i\omega} \frac{\Gamma \left( \frac{i\gamma \omega}{2\pi} \right) \Gamma \left( \frac{1}{2} + \frac{i\omega}{2} \right) \Gamma \left( -\frac{\pi + \gamma}{2\pi} \omega \right)}{\Gamma \left( \frac{-i\gamma \omega}{2\pi} \right) \Gamma \left( \frac{1}{2} - \frac{i\omega}{2} \right) \Gamma \left( \frac{\pi + \gamma}{2\pi} \omega \right)} (4.84)$$

and $C_+$ again encircles the singularities of $\rho(\omega)$ on the positive imaginary axis. Here we have the “cosine” poles at

$$\omega = 2i\pi n/\gamma, \quad n = 1, 2, \ldots$$

$$c_n = \frac{i^2}{2\pi} e^{4i\pi n/\gamma} \text{res}_{\omega=2i\pi n/\gamma} \rho(\omega) = \frac{(-)^n}{n!(n-1)!} \frac{\Gamma \left( \frac{1}{2} - \frac{\pi n}{\gamma} \right) \Gamma \left( \frac{\pi + \gamma}{\gamma} n \right)}{\Gamma \left( \frac{1}{2} + \frac{\pi n}{\gamma} \right) \Gamma \left( \frac{-\pi + \gamma}{\gamma} n \right)} (4.85)$$

along with the “vacuum” ones

$$\omega = i(2n + 1), \quad n = 0, 1, 2, \ldots$$

$$s_n = \frac{i}{2} e^{2\gamma (2n+1)} \text{res}_{\omega=(2n+1)i} \rho(\omega)$$

$$= \frac{(-)^n}{(n!)^2} \frac{\Gamma \left( \frac{- (2n+1) \gamma}{2\pi} \right) \Gamma \left( \frac{(\pi + \gamma)(2n + 1)}{2\pi} \right) \Gamma \left( \frac{(\pi + \gamma)(2n + 1)}{2\pi} \right)}{\Gamma \left( \frac{- (2n+1) \gamma}{2\pi} \right) \Gamma \left( \frac{(\pi + \gamma)(2n + 1)}{2\pi} \right) \Gamma \left( \frac{(\pi + \gamma)(2n + 1)}{2\pi} \right)} (4.86)$$

The function $w(\omega)$ is regular in the upper half-plane. If $\gamma > 2\pi$ (i.e. $\lambda < 1/3$)
the leading correction to (4.80) is due to the first "cosine" pole at $\omega = 2i\pi/\gamma$

$$\mathcal{F}^{(-)}(H) = -\frac{(1 - \lambda)H^2}{\pi} \times \left(1 + 2\frac{(1 - \lambda)^2}{(1 + \lambda)^2} c_1(\lambda) \left[\frac{4H\lambda \Gamma\left(\frac{1}{2\lambda} + \frac{1}{2}\right)}{M\sqrt{\pi} \Gamma\left(\frac{1}{2\lambda}\right)}\right]^{4\lambda/(1 - \lambda)} + \ldots\right)$$

$$c_1(\lambda) = -\frac{\lambda}{(1 - \lambda)^2} \frac{\Gamma\left(\frac{1 - 3\lambda}{2 - 2\lambda}\right) \Gamma\left(\frac{1}{1 - \lambda}\right)}{\Gamma\left(\frac{1 + \lambda}{2 - 2\lambda}\right) \Gamma\left(\frac{1 - 2\lambda}{1 - \lambda}\right)}$$

This deviation has to be compared with the second-order irrelevant perturbation in the effective action (1.34)

$$F^{(-)}(H) - F^{(-)}(0) = -\frac{(1 - \lambda)H^2}{\pi} \times \left(1 + (1 - \lambda) \frac{\Gamma\left(-\frac{1}{1 - \lambda}\right)}{\Gamma\left(\frac{2}{1 - \lambda}\right)} \frac{M_0^4}{64} (2H)^{4\lambda/(1 - \lambda)} + \ldots\right)$$

As a byproduct we obtain the following exact relation between the irrelevant coupling $M_0$ in (1.34) and the intercept mass scale $M$:

$$\frac{M_0^2}{16} = \frac{\Gamma\left(\frac{1}{1 - \lambda}\right)}{\Gamma\left(-\frac{1}{1 - \lambda}\right)} \left[2(1 - \lambda) \frac{\Gamma\left(\frac{1 - 2\lambda}{1 - \lambda}\right)}{M\sqrt{\pi} \Gamma\left(\frac{1}{2\lambda}\right)}\right]^{2\lambda/(1 - \lambda)}. \quad (4.90)$$

For $\pi < \gamma < 2\pi$ ($1/3 < \lambda < 1/2$) the first "vacuum" pole at $\omega = i$ becomes most important in the IR asymptotic. The corresponding contribution is

$$\mathcal{F}^{(-)}(H) = -\frac{(1 - \lambda)H^2}{\pi} \left(1 + \frac{1 - \lambda}{\pi} \tan\frac{\pi}{2\lambda} \left(\frac{2H}{M}\right)^2 + \ldots\right). \quad (4.91)$$

In terms of the IR effective action this behavior can be attributed to the first counterterm omitted in (1.34)

$$\mathcal{A}_{\text{IR}} = \frac{1}{4\pi} \int \left[(\partial_\mu \chi)^2 + \frac{M_0^2}{2} \alpha^2 \cos\frac{2}{\alpha\chi} - G\bar{T}T + \ldots\right] d^2\chi,$$  \quad (4.92)
"vacuum" counterterm contributes just (4.91) if
\[ G = -\frac{16}{\pi M^2} \tan \frac{\pi}{2\lambda} . \] (4.93)

The asymptotics (4.88) and (4.91) are related to the plane IR limit \( H \to 0 \) (with \( \lambda \) fixed) where only one leading pole contributes. In the IR scaling region \( H \ll M, \lambda \ll 1 \) while \( \lambda \log(M/H) \approx 1 \) one has to iterate eq. (4.84) through all the "cosine" poles (ignoring the vacuum ones). This results in the following "loop expansion":
\[ \mathcal{F}^{(-)}(H) = -\frac{(1-\lambda)^2 H^2}{\pi} \times \left[ 1 + \lambda \frac{1+q}{1-q} + \lambda^2 \left( \frac{1+q}{1-q} \right)^2 - \frac{4\lambda^2 q}{(1-q)^2} \log \frac{1-q}{4\lambda} + O(\lambda^3 \log^2 \lambda) \right], \] (4.94)

where we have liberated \( q \) from (4.56) to use it as
\[ q = \left[ \frac{2^{3/2} H}{e^{3/4} \sqrt{\pi} M} \right]^{4\lambda/(1-\lambda)} , \] (4.95)

Note that the "one-loop" term \( \lambda (1+q)/(1-q) \) of expansion (4.94) is quite what one has in the perturbation theory of the IR action (1.36)
\[ F^{(-)}(H) - F^{(-)}(0) = -\frac{\hbar^2}{4\pi} \left( 1 - \frac{f}{2} + O(f^2) \right) \] (4.96)

with the one-loop flow (1.39). We have also verified that the two-loop perturbative correction to (1.37) and (4.96) agrees perfectly with the next terms in (4.94). In particular at \( \lambda = 0 \) (the O(3) SM at \( \theta = \pi \)) this loop expansion becomes
\[ F^{(-)}(H) - F^{(-)}(0) = -\frac{\lambda^2}{4\pi} \left( 1 + \frac{1}{2 \log(A'/A)} - \frac{\log \log(A'/A)}{4 \log^2(A'/A)} \right) \] (4.97)

with the \( A' \) parameter related to the physical scale \( M \) as follows:
\[ A' = \frac{e^{1/4} \sqrt{\pi}}{2^{3/2}} M . \] (4.98)

5. Sausage at finite temperature (the hot sausage)

Here we perform another test by putting our sausage model at a finite temperature \( T \). The short-distance behavior occurs at high temperatures where we are going to compare the perturbative predictions of the SSM\( \theta \) lagrangian (3.10)
with what follows from the scattering theories of sect. 2. The high-temperature information that we are able to obtain analytically from the FST data is much less detailed than in the previous case of external gauge field. For this reason we mainly apply a numerical analysis. However, the UV behavior (which is crucial for the local field theory interpretation) turns out to be very non-trivial and we believe that the good numerical agreement demonstrated below cannot be accidental.

In the finite-temperature settlement one considers a two-dimensional relativistic euclidean field theory at an infinite flat cylinder of circumference $r = l/T$. The main object we address in this section is the finite-temperature specific free energy $f(T)$. In the relativistic case one can equally consider this observable as the ground-state energy $E(r) = f(T)/T$ of the same field theory living on a finite space circle (of length $r$). The first (thermodynamic) point of view is preferable in the S-matrix approach while the second (finite-size) one is more convenient when considering the lagrangian framework.

If we have a unitary field theory which tends at short distances to some UV CFT the leading $r \to 0$ behavior of the ground-state energy is simply related to the corresponding UV conformal central charge $E(r) \sim -\pi c_{UV}/6r$ [36]. It is convenient therefore to introduce the effective central charge $c(r)$ as

$$E(r) = -\frac{\pi c(r)}{6r}$$

so that

$$c(0) = c_{UV}.$$

After these well-known generalities let us turn to the SSM($\theta$) (3.10) in the cylinder geometry and denote by $c_{\nu}^{(\theta)}(r)$ the corresponding effective central charge. Taking for the moment the finite size point of view we call $x_1$ the “space” coordinate along the basic circle ($x_1 \sim x_1 + r$) and $x_2$ the “imaginary time” along the cylinder. At small $r$ it is very natural to separate the fields $X, Y$ into the $x_1$-independent “zero mode” coordinates $x, y$

$$X(x_1, x_2) = x(x_2) + \xi(x_1, x_2),$$

$$Y(x_1, x_2) = y(x_2) + \eta(x_1, x_2)$$

and the oscillations $\xi(x_1, x_2), \eta(x_1, x_2)$ around the “center of mass” such that $\int \xi dx_1 = \int \eta dx_1 = 0$. In the finite-size geometry these oscillations have two effects. First they produce the well-known conformal anomaly with the “Goldstone” UV central charge

$$c_{UV} = 2.$$  

The second effect of the oscillations $\xi, \eta$ is the renormalization of the SM metric which follows the RG evolution of eq. (1.6). After the oscillating modes are integrated out one is left with the quantum mechanics of the “zero-mode
particle" moving in the renormalized (at scale $t = \log r$) target space. The corresponding hamiltonian is

$$\mathcal{H} = \frac{1}{2t} e^{-\Phi(y)} \left[ P_x^2 + P_y^2 \right], \quad (5.5)$$

where $P_x$ and $P_y$ are the momenta canonically conjugated to the zero-mode coordinates $x, y$. Although it is quite well known that two-dimensional field theory at high temperature is reduced typically to a quantum mechanical problem, eq. (5.5) suffers from the standard problem of the quantum mechanics on curved manifolds, i.e. the ordering ambiguity. Definitely the high-temperature limit of two-dimensional sigma model prescribes some particular ordering in (5.5). As far as we know this problem has not been addressed in literature. It looks most natural to take the completely symmetric (Weyl) ordering (we denote it by the symbol S), i.e. we propose that the quantum hamiltonian

$$\hat{\mathcal{H}} = -\frac{1}{2t} S \left[ e^{-\Phi(y)} \left( \partial_x^2 + \partial_y^2 \right) \right] \quad (5.6)$$

correctly describes (at least to the one-loop approximation) the zero-mode dynamics in the high-temperature SM. Note that eq. (5.6) is a suggestion and its field theoretic proof remains an open problem.

Operator (5.6) can be readily evaluated using the following simple formula:

$$S \left( F(y) \partial_y^2 \right) = F(y) \partial_y^2 + F'(y) \partial_y + \frac{1}{2} F''(y). \quad (5.7)$$

We find

$$\hat{\mathcal{H}} = \frac{1}{r} e^{\Phi(y)/2} \hat{h} e^{-\Phi(y)/2} \quad (5.8)$$

where $\hat{h}$ is self-conjugate with respect to the natural scalar product in the SM metric

$$(\Psi_1, \Psi_2) = \int \Psi_1^* \Psi_2 e^{\Phi(y)} \, dx \, dy \quad (5.9)$$

and has the following covariant form:

$$\hat{h} = -\frac{1}{2} \nabla^2 + \frac{1}{8} R. \quad (5.10)$$

Here $\nabla^2 = e^{-\Phi} (\partial_y^2 + \partial_x^2)$ is the Laplace operator in the SM metric and $R = -e^{-\Phi} \partial_y^2 \Phi$ is its scalar curvature.

The effective central charge $c^{(\theta)}_\nu (r)$ becomes

$$c^{(\theta)}_\nu (r) = 2 - e_\nu (r) + \ldots, \quad (5.11)$$

where the one-loop correction $e_\nu (r)$ is the lowest eigenvalue of the operator $\hat{h}$

$$\hat{h} \Psi_0 = \frac{\pi e_\nu}{6} \Psi_0. \quad (5.12)$$

In the actual axially symmetric geometry the ground-state wave function $\Psi_0$ is obviously independent on the azimuthal coordinate $x$ and eq. (5.12) acquires
the form
\[-\frac{1}{2} \psi''(y) - \frac{1}{8} \Phi''(y) \psi_0(y) = e^{\Phi(y)} \frac{\pi \epsilon v}{6} \psi_0(y). \tag{5.13}\]

The renormalized metric at scale \(r\) corresponds to
\[t - t_0 = \log r A_0, \tag{5.14}\]
where \(A_0\) is the normalization parameter. Note that with the one-loop metric (3.4), (3.7) the differential operator \(h/\nu\) depends on the scaling combination \(\eta = \nu (t_0 - t)/4\pi\) only. Therefore
\[e_\nu(r) = \frac{\nu \kappa(\eta)}{4\pi}, \tag{5.15}\]
where the scaling function \(\kappa(\eta)\) appears as the minimum eigenvalue of the Sturm–Liouville problem
\[-(\cosh 2\eta + \cosh 2y) \partial_y^2 \psi_0 + \frac{1 + \cosh 2\eta \cosh 2y}{\cosh 2\eta + \cosh 2y} \psi_0 = \frac{\kappa}{6} \sinh 2\eta \psi_0, \tag{5.16}\]
The substitution
\[e^{\nu - \eta} = \frac{\text{cn}(v|k^2)}{\text{sn}(v|k^2)}, \]
\[\psi_0 = \sqrt{\frac{\text{sn}(v|k^2) \text{cn}(v|k^2)}{\text{dn}(v|k^2)}} \psi_0 \tag{5.17}\]
where \(k^2 = 1 - \exp(-4\eta)\), brings eq. (5.16) to the Lamé form
\[-\partial_v^2 \psi_0 - \frac{\text{cn}^2(2v|k^2)}{\text{sn}^2(2v|k^2)} \psi_0 = \frac{\kappa(\eta)}{6} k^2 \psi_0, \quad 0 < v < K \tag{5.18}\]
with the boundary conditions
\[\psi \sim \begin{cases} \sqrt{v} & \text{at } v \to 0 \\ \sqrt{K - v} & \text{at } v \to K \end{cases} \tag{5.19}\]
Here \(K = K(k^2)\) is the real period of the elliptic functions involved. It is interesting to note that eq. (5.18) also can be reduced to the Fuchs-type equation with four parabolic singular points, the parameter \(\kappa\) becoming the corresponding accessor coefficient. Equations of this type were studied in the classical theory of Liouville equation.

Eq. (5.18) can be considered as an elliptic deformation of the differential equation for the Legendre polynomials. It can be studied analytically in two limits: \(k^2 = 1 - \exp(-4\eta) \to 0 (\eta \to 0)\) and \(k^2 \to 1 (\eta \to \infty)\). In the first case \(\eta \to 0\) the eigenvalues can be developed by the standard perturbation theory near the exact solutions \(\psi_l(v) = \sqrt{\sin 2v} P_l(\cos 2v)\), where \(P_l\) are the Legendre polynomials. For the lowest eigenvalue \(\kappa\) we find the following expansion:
\[\kappa(\eta) = \frac{3}{\eta} - \frac{4}{45} \eta^3 + \frac{8}{5 \cdot 7 \cdot 9^2} \eta^5 + O(\eta^7). \tag{5.20}\]
In the opposite limit $\eta \to \infty$, $k^2 = 1 - \exp(-4\eta) \to 1$, the real period $K \sim \frac{1}{2} \log 16/(1 - k^2) \sim 2\eta + 2\log 2 \to \infty$. In this case the potential term in eq. (5.18) can be approximated as

$$\frac{-\text{cn}^2(2v|k^2)}{\text{sn}^2(2v|k^2)} = \begin{cases} -1/\sinh^2 2v & 0 < v \ll K \\ -1/\sinh^2 2(K - v) & 0 < K - v \ll K \end{cases}$$

(5.21)

In the central region $v \gg 1$, $K - v \gg 1$ one can neglect the potential term and take

$$\psi = \cos p(v - K/2)$$

(5.22)

where $\kappa = 6p^2$. Matching together (5.22) and the exact solution in the potential (5.21) (with the boundary conditions (5.19)) one finds that $p = \pi/(K + 2\log 2) \sim \pi/(2\eta + 4\log 2)$ and therefore

$$\kappa = 6p^2 = \frac{3\pi^2}{2(\eta + 2\log 2)^2} + O(\eta^{-4}).$$

(5.23)

Eqs. (5.19) and (5.22) define the asymptotics of function $\kappa(\eta)$ for $\eta \ll 1$ and $\eta \gg 1$. In the intermediate region $\kappa(\eta)$ can be computed numerically. This work has been performed in ref. [37]. Below we use the numerical data to compare with what follows from the scattering theories of sect. 2.

Naturally, beyond the scaling (one-loop) correction (5.15) we expect some systematic higher-loop expansion of the effective central charge (5.11). To evaluate these terms one has to take into account the higher-loop corrections to the SM metric trajectory (3.4), (3.7) together with possible higher-loop modifications of the zero-mode dynamics (5.10).

In the perturbative considerations above we have seen no dependence on the topological angle $\theta$. It appears in the instanton contributions to the ground-state energy $E_\nu^{(0)}$. Any $q$-instanton term with $q \neq 0$ is invisible in the perturbation theory being suppressed as $\exp 2|q|t$ at $t \to -\infty$. The leading $\theta$-dependent correction is proportional to $\cos \theta$ and comes from the one-instanton configuration, which develops the standard “small-instanton” logarithmic divergency. Apparently this divergence contributes in the same way as to the bulk vacuum energy. For two integrable cases $\theta = 0$ and $\theta = \pi$ the prefactor can be borrowed from the result of sect. 4 eq. (4.54). In terms of physical mass parameters $M = m$

$$E^{(0,\pi)}_{\text{one-inst}} = \mp \frac{m^2r}{4\pi} (\log mr + \text{const.})$$

(5.24)

This one-instanton logarithmic contribution determines the leading asymptotic in the difference $E^{(0)}_\nu(r) - E^{(\pi)}_\nu(r)$.

Now let us turn to the factorized scattering language and consider the same physical observable $E(r)$, i.e. the finite-size ground-state energy, as it follows from the FST’s SST$^+_{\kappa}$ of sect. 2. We implement the standard thermodynamic Bethe ansatz (TBA) technique [26,27] reducing the problem to a system of
non-linear integral equations. For general irrational \( \lambda \) this is an infinite system of coupled non-linear equations, which becomes finite at \( \lambda \) rational \(^{[38]}\). The system is particularly simple if

\[
\lambda = 1/N
\]

(5.25)

with \( N = 2, 3, \ldots \) integer. This particular set of values is sufficient for our purposes since we are able to reach the scaling region if \( N \) is large enough. For instance, in the limit \( N = \infty \) we are at the \( O(3) \) symmetric point.

It turns out that the two apparently different FST’s SST\(_\lambda^+\) and SST\(_\lambda^-\) result (at the same \( \lambda = 1/N \)) in very similar TBA systems. At \( N \geq 3 \) we have a common system of \( N + 1 \) coupled non-linear integral equations for \( N + 1 \) functions \( \varepsilon_a(\beta), a = 0, 1, 2, \ldots, N \) (called the pseudo energies) of the rapidity variable \(-\infty < \beta < \infty\). The TBA system has the following form:

\[
\rho_a(\beta) = \varepsilon_a(\beta) + \frac{1}{2\pi} \int \sum_{b=0}^{N} \phi_{ab}(\beta - \beta')L_b(\beta') d\beta'
\]

(5.26)

where \( L_a(\beta) = \log(1 + \exp(-\varepsilon_a(\beta))) \) and

\[
\phi_{ab}(\beta) = \frac{l_{ab}}{\cosh \beta}.
\]

(5.27)

In the last equation \( l_{ab} \) is the incidence matrix of the extended (affine) \( D_N \) Dynkin diagram (see fig. 6). The difference between SST\(_\lambda^+\) and SST\(_\lambda^-\) is only in the “energy terms” \( \rho_a(\beta) \). For SST\(_\lambda^+\) we have

\[
\rho_a(\beta) = m\delta_{0a} \cosh \beta
\]

(5.28)

as it is indicated in fig. 6a, while in the case of massless SST\(_\lambda^-\) (fig. 6b)

\[
\rho_a(\beta) = \frac{m\delta_{0a}}{2}e^\beta + \frac{m\delta_{1a}}{2}e^{-\beta}.
\]

(5.29)

The finite-size ground-state energy \( E^{(\pm)}(r) \) reads in both cases

\[
E^{(\pm)}(r) = -\frac{1}{2\pi} \int \sum_a \rho_a^{(\pm)}(\beta)L_a(\beta) d\beta.
\]

(5.30)

In the limit \( N = \infty \) we recognize the TBA systems for the \( O(3) \) SM at \( \theta = 0 \) and \( \pi \) suggested in ref. \([31]\).

First we consider the short distance \( (r \rightarrow 0) \) behavior of \( E^{(\pm)}(r) \) predicted by the TBA systems (5.26)–(5.30). It is convenient to introduce again the effective central charges \( c^{(\pm)}(r) = -6rE^{(\pm)}/\pi \). Then the standard dilogarithm calculation \([26, 27]\) gives

\[
\lim_{r \rightarrow 0} c^{(\pm)}(r) = 2
\]

(5.31)

independently on \( \lambda \). This is just the “Goldstone” UV central charge (5.4) we have already learned from the perturbation theory of the sausage model. Let us
define like in eq. (5.11)
\[ e^{(\pm)}(r) = 2 - c^{(\pm)}(r). \]  
(5.32)

The analytic structure of this deviation turns out much more complicated then it is usual in integrable perturbed CFT’s [26,27] and rather resembles that of the sinh–Gordon model [39]. The leading asymptotic at \( \tau = -\log m r \to \infty \) (\( N \) fixed) can be analysed along the lines of ref. [39]. At \( \tau \gg 1 \) the functions \( L_a \) in eq. (5.26) become large in the “central region” \( |\beta| \lesssim \tau \) and behave as (independently on the choice of the energy terms (5.28) or (5.29))

\[
L_0 \simeq L_1 \simeq L_{N-1} \simeq L_N = L, \\
L_a \simeq 2 \left( L - \frac{(a-1)(N-1-a)}{N-2} e^{-L} \right), \quad a = 2, \ldots, N-2, 
\]  
(5.33)

where
\[
L = 2 \log \left( \frac{4 \tau \cos \pi \beta / (2 \tau)}{\pi^2 (N-2)^{1/2}} \right). \]  
(5.34)

Appropriate calculation (similar to that of ref. [39]) leads to the following asymptotic
\[
e^{(\pm)}(\tau) \simeq \frac{3 \pi^2 (N-2)}{2 \tau^2} + O \left( \frac{1}{\tau^3} \right). \]  
(5.35)

In principle there is an indirect way to estimate another interesting limit \( N \to \infty, N/\tau \to \infty \), while \( \tau \gg 1 \). We use the following observation. This limit corresponds to the UV asymptotics of the O(3) SM. As it was shown in ref. [31] the O(3) SM in turn can be considered as the \( n \to \infty \) limit of some integrable perturbations of the \( \mathbb{Z}_n \) parafermionic CFT’s. Denote \( \delta_{PF}(r, n) \) the corresponding deviations (defined as in eq. (5.32)) in these perturbed parafermionic theories.
Then \( \lim_{n \to \infty} e_{PF}(r, n) = \lim_{N \to \infty} e^{(\pm)}(r, N) \). The short-distance structure of \( e_{PF} \) was considered in ref. [31]. It is more regular than that of \( e^{(\pm)}(r) \) and roughly has the following form:

\[
e_{PF}(r, n) = \frac{6}{n + 2} + \sum_{k=1}^{\infty} b_k(n) e^{-4k\tau/n} \tag{5.36}
\]

with some coefficients \( b_k(n) \). At \( \tau \gg 1 \) the main contribution comes from the terms of order \( 0 < k < n/\tau \). Appropriate analysis of the coefficients \( b_k(n) \) at large \( n \) shows that in this region of \( k \)

\[
b_k(n) = \frac{12}{n} (1 + O(k \log(k/n)/n)) \tag{5.37}
\]

so that

\[
e_{PF}(r, n) = \frac{6(1 + e^{-4\tau/n})}{n(1 - e^{-4\tau/n})} + O(\log n/n) . \tag{5.38}
\]

In the limit \( n \to \infty \) we have

\[
\lim_{n \to \infty} e^{(\pm)} = \lim_{N \to \infty} e_{PF} = \frac{3}{\tau} + O(\log \tau/\tau^2) . \tag{5.39}
\]

Both asymptotics (5.35) and (5.39) are consistent with the following scaling behavior in the limit \( N \to \infty, \tau \to \infty \) while \( \eta = \tau/N \) is fixed

\[
e^{(\pm)} = \kappa(\eta)/N = \lambda \kappa(\eta) \tag{5.40}
\]

The relation (4.29) taken into account, eq. (5.40) is quite the same as the corresponding scaling form (5.15) we have observed perturbatively in the sausage model. The asymptotics (5.35) and (5.39) of the scaling function are in agreement with the corresponding limits (5.23) and (5.20). We suppose that the two scaling functions defined in eq. (5.15) (in terms of the eigenvalue problem) and (5.40) (related to the TBA system) are in fact the same, so that we use the same notation for them.

We have verified the scaling behavior (5.40) by numerical solution of two TBA systems (5.26–30) for \( N \) in the range between 21 and 65. The corresponding numerical data (which do not distinguish in this limit between \( e^{(+) \pm} \)) are plotted in figs. 7 and 8 together with the numerical solution of the eigenvalue problem (5.18) [37]. One can see an impressive numerical agreement.

We have also analysed numerically the difference \( e^{(+) \pm}(r) - e^{(-) \pm}(r) \) which was estimated above by the instanton arguments. This difference occurs to be (in the leading behavior at \( r \to 0 \)) independent on \( N \) and in perfect numerical agreement with (5.24).

Being very small at \( mr \ll 1 \) the difference \( e^{(+) \pm}(r) - e^{(-) \pm}(r) \) becomes substantial at \( mr \sim 1 \) and finally at \( mr \gg 1 \) the central charges \( c^{(+)}(r) \) and \( c^{(-)}(r) \) behave in completely different way. While for \( E^{(+)}(r) = -\pi c^{(+)}(r)/6r \) we
expect a characteristic massive exponential falloff
\[ E^{(+)}(r) = -\frac{3m}{2\pi} \int e^{-mr \cosh \beta} \cosh \beta \, d\beta + O(\exp(-2mr)), \quad (5.41) \]
the massless SST_{\lambda}^{(-)} develops a non-trivial IR structure. In the IR limit \( r \to \infty \) the effective central charge \( c^{(-)}(r) \) tends to the value
\[ \lim_{r \to \infty} c^{(-)}(r) = c_{\text{IR}} = 1 \quad (5.42) \]
independently on \( \lambda = 1/N \). This is exactly what is implied by our identification of the family U(1)_{a} as the IR CFT of the sausage model at \( \theta = \pi \). Moreover, if \( N \) is large enough one can apply the periodicity arguments (like that of ref. [26]) to argue the following form of the IR corrections to the limiting value (5.42):
\[ c^{(-)}(r, N) = 1 + d_{1}(N)(mr)^{-4/(N-1)} + d_{2}(N)(mr)^{-8/(N-1)} + \ldots \quad (5.43) \]
with some coefficients \( d_{n}(N) \). At finite \( N \) this asymptotic expansion includes
also some corrections of different analytic structure (see below). Behavior (5.43) is similar to the infrared expansion (4.89) of the vacuum energy in external magnetic field and quite meets our interpretation of (1.34) (with $\Delta_{\text{IR}} = \alpha^{-2} = 1/(1 - \lambda)$ as the effective IR action of $\text{SSM}_{\nu}^{(\pi)}$).

Note that since we know the relations (4.90) and (4.93) between the IR perturbation couplings and the physical intercept scale $M = m$, we are able to calculate the leading IR corrections in closed form. Thus, doing the second-order
IR perturbation in \(M_0^2\) (which is leading at \(\lambda < 1/3\)) we find
\[
c^{(-)}(r) = 1 + \frac{12\lambda}{(1 - \lambda)(1 + \lambda)^2} \Gamma^4 \left( \frac{1}{1 - \lambda} \right) \Gamma \left( \frac{1 - 3\lambda}{1 - \lambda} \right) \]
\[
\times \left[ \frac{4(1 - \lambda)\sqrt{\pi}}{mr} \frac{\Gamma \left( \frac{1 - \lambda}{2\lambda} \right)}{\Gamma \left( \frac{1}{2\lambda} \right)} \right]^{4i/(1 - \lambda)} + \ldots \quad (5.44)
\]
In particular we have an exact prediction for the coefficient \(d_1(N)\) in (5.43)
\[
d_1(N) = \frac{12}{(N - 1)(N + 1)^2} \frac{\Gamma^4 \left( \frac{N}{N - 1} \right) \Gamma \left( \frac{N - 3}{N - 1} \right)}{\Gamma \left( \frac{N - 2}{N - 1} \right) \Gamma \left( \frac{N + 1}{N - 1} \right)}
\times \left[ \frac{4(N - 1)\sqrt{\pi}}{N} \frac{\Gamma \left( \frac{N - 1}{2} \right)}{\Gamma \left( \frac{N}{2} \right)} \right]^{4/(N - 1)}. \quad (5.45)
\]
In the region \(1/3 < \lambda < 1/2\) the \(G\)-perturbation in (4.92) becomes leading and we have
\[
c^{(-)}(r) = 1 + \frac{2\pi}{3(mr)^2} \tan \frac{\pi}{2\lambda} + O \left( (mr)^{4i/(1 - \lambda)} \right). \quad (5.46)
\]
At \(\lambda = 1/3\) the two corrections (5.44) and (5.46) interfere producing a logarithmic contribution and the effective central charge behaves as
\[
c^{(-)}(r) = 1 + \frac{4}{3(mr)^2} \left( \log mr + \log 3 + 2\log 2 + C - \frac{11}{3} \right)
\]
\[
+ O \left( \frac{\log mr}{(mr)^4} \right), \quad \lambda = \frac{1}{3}. \quad (5.47)
\]
where \(C = 0.577216\ldots\) is Euler’s constant. Note that this value of \(\lambda\) corresponds to \(N = 3\) so that (5.47) is our prediction for the IR behavior of the “square” TBA system in fig. 6b.

The IR effective action in the form (1.36) is more convenient to estimate the leading asymptotic of \(c^{(-)}(r)\) in the scaling limit \(\lambda \to 0\), \(\log mr \to \infty\), \(\lambda \log mr\) fixed. The first correction appears at the third order in \(f\) and we have
\[
c^{(-)}(r) = 1 - \frac{3}{8} f_{\parallel}(t) f_{\perp}^2(t) + O(f^4), \quad (5.48)
\]
where \(f_{\parallel}(t)\) and \(f_{\perp}(t)\) follow (to the one-loop approximation) the flow (1.39) with \(t - t_0 \simeq \log rm\) and \(c = 2(\alpha^2 - 1) = 2\lambda + O(\lambda^2)\).

We have verified all these IR predictions by solving numerically the SST\(_{\lambda}^{(-)}\) TBA system at large \(mr\).
6. Concluding remarks

Above we have considered the "inverse scattering problem" for a particular SM, i.e. the sausage model (3.10). We believe that the sausage trajectory we have found in sect. 3 is the one-loop approximation to some integrable deformation of the two-dimensional sphere. On the other hand at present several parametric families of FST's are known which at some special points are reduced to the FST's of integrable SM's. It seems like true that every such situation is related to some family of RG trajectories corresponding to integrable deformations of known (typically rather symmetric) manifolds.

As we have seen in the last section some important and very non-trivial information about the RG metric trajectory $G_{ij}(t)$ is encoded in the high-temperature behavior of the effective central charge. For the SM’s with two-dimensional target space we propose eqs. (5.10)--(5.12) as the correct one-loop description of the short-distance asymptotic. There are many reasons to believe that these equations are of more general nature, being relevant for all two-dimensional SM's at high temperatures. In particular we have verified this conjecture for the $O(N)$ SM’s in the 1/$N$ expansion. It is relevant to note that operator (5.10) has been considered in ref. [40] in relation to the quantum mechanics on Riemann manifolds.

However, the problem of higher-loop corrections requires some further study. It is clear in principle how to develop the loop expansion for the SM metric evolution (see e.g. ref. [3]). The zero-mode dynamics of eq. (5.10) may also be modified after the higher-loop corrections are properly taken into account. Does it persist a quantum mechanical problem at higher perturbative orders? The nature of possible modifications remains to be found together with a systematic way of their evaluation.

There is another problem closely related to that outlined above. In sects. 2 and 3 we have actually settled two different formulations of the sausage sigma model. The first one of sect. 2 is based on the scattering theories SST$_{\lambda}^{(\pm)}$ and involves their parameter $\lambda$. The second definition (sect. 3) is essentially perturbative so that the corresponding parameter $\nu$ of the SM trajectory is not yet specified beyond the one-loop order. Let us suppose that there is an exact UV-stable metric trajectory which solves the RG equation (1.6) to all orders and bears qualitatively the same properties as the one-loop one, i.e. in the UV limit $t \to -\infty$ it becomes a long almost flat cylinder with uniformly and monotonously expanding round ends. Then we are able to prescribe an exact meaning to the parameter $\nu$ (even if it is not small) say through the relation (3.9) where $l$ is the asymptotic circumference of the intermediate cylinder.

At this point one could find it desirable to have an exact (or at least higher-loop) relation between these two (now precisely specified) parameters $\lambda$ and $\nu$. So far it was determined only up to the leading one-loop order (eq. (4.29)).
In particular this relation would provide us with exact values of the charges $Q^{(±)}(λ)$ introduced in sect. 4, since, as it follows from eqs. (4.6) and (4.28),
\[
\left[Q^{(+)}(λ)\right]^2 = \frac{4πλ}{ν}.
\] (6.1)

Note that in order to perform this program we have no need to solve exactly for the whole RG trajectory of the sausage SM. It would be sufficient to possess the exact (up to all loop orders) Witten's euclidean black-hole metric which is supposed to be the asymptotic shape of the sausage ends. For, at $t \to -∞$ the curvature in the cylinder part of the sausage is always small independently on $ν$ and one can safely use the one-loop equation (3.1). A suitable solution would be (instead of (3.4), (3.7))
\[
e^{-φ(γ,t)} = \frac{1}{2} (A(t) + B(t)\cosh(2σY)) ,
\] (6.2)
where at $t \to -∞$
\[
A(t) \approx ν ,
\]
\[
B(t) \approx 2ν \exp\left(-\frac{σ^2ν}{2π} (t_0 - t)\right)
\] (6.3)
with some parameter $σ$ to be determined. Metric (6.2) is only valid in the “cylinder region” $|Y| \lesssim σν(t_0 - t)/4π$ where the curvature is small. Now we are not bound to respect the condition (3.2) and parameter $σ$ is rather determined in the “transition region” $Y \sim σν(t_0 - t)/4π$ where the one-loop metric (6.2) appears as
\[
e^{-φ(γ,t)} = ν \left(1 + \exp\left(26 \left[Y - \frac{σν}{4π} (t_0 - t)\right]\right)\right)
\] (6.4)
and can be considered as the “infinite cylinder” asymptotic of the exact black-hole solution $\exp(-φ_{bh}(Y + σνt/4π))$. Here $φ_{bh}(Y)$ is supposed to solve (to all loop orders) the fixed-point RG equation
\[
\frac{σν}{4π} ∂_Y φ = \frac{1}{4π} R + \frac{1}{(4π)^2} R^2 + \ldots
\] (6.5)
with the boundary conditions
\[
exp(-φ(Y)) \to ν \quad \text{at} \quad Y \to -∞
\]
\[
exp(-φ(Y)) \sim exp(2Y) \quad \text{at} \quad Y \to ∞
\] (6.6)

In ref. [41] an exact black-hole metric was proposed. Taking it seriously one would immediately find
\[
σ = \sqrt{\frac{k}{k - 2}} ,
\] (6.7)
where in terms of $ν$ the WZW parameter $k$ of ref. [41] reads
\[
k = \frac{4π}{ν} .
\] (6.8)
This in turn results in the following relation
\[ \frac{1}{k} = \frac{\nu}{4\pi} = \frac{\lambda}{1 + \lambda} \]  \hspace{1cm} (6.9)
and therefore
\[ Q^{(+)} = \sqrt{1 + \lambda} . \] \hspace{1cm} (6.10)

Unfortunately the metric of [41] is only a conjecture and it is not clear if it really solves eq. (6.5) at all loop orders. We have verified that it does up to two loops. Although we keep some doubts that it works beyond, the two-loop result implies that at least
\[ \frac{\nu}{4\pi} = \lambda - \lambda^2 + O(\lambda^3) \] \hspace{1cm} (6.11)
and
\[ Q^{(+)} = 1 + \lambda/2 + O(\lambda^2) \] \hspace{1cm} (6.12)
independently on the guess of ref. [41].

As it was conjectured by Haldane, the O(3) SM at $\theta = 0 \ (\theta = \pi)$ describes the continuum limit of the SU(2) invariant Heisenberg spin chain (the XXX model) with large integer (half-integer) spin. We think that the sausage model is related to the similar large-spin asymptotics of anisotropic Heisenberg spin chain with U(1) symmetry (i.e. the higher-spin XXY model). This hypothesis can be checked by comparing the thermodynamic or magnetic functions of the large-spin anisotropic chains with the corresponding quantities in the sausage model. However, the relation between microscopic parameters of a particular XXY chain near criticality and our universality parameters ($\lambda$ and $m$) remains to be established.

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References

E. Abdalla, M.B. Abdalla and K.P Rothe, 2-Dimensional quantum field theory (World Scientific, Singapore, 1991)
Kiev, 1988;
M. Takahashi, Prog. Theor. Phys. 46 (1971) 401
[28] A. Belavin and A. Polyakov, JETP Lett. 22 (1975) 503
[37] R. Brunelli and G.P. Teclhioi, Stochastic minimization with adaptive memory, IRST
preprint, sept. 1992, Pove, Trento (Italy)