Expectation values of descendent fields in the sine-Gordon model

Vladimir Fateev \textsuperscript{a,c}, Dmitri Fradkin \textsuperscript{b}, Sergei Lukyanov \textsuperscript{b,c,1}, Alexander Zamolodchikov \textsuperscript{b,c}, Alexei Zamolodchikov \textsuperscript{a}

\textsuperscript{a} Laboratoire de Physique Mathématique, Université de Montpellier II, Pl. E. Bataillon, 34095 Montpellier, France
\textsuperscript{b} Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA
\textsuperscript{c} L.D. Landau Institute for Theoretical Physics, Chernogolovka, 142432, Russia

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Abstract

We obtain exactly the vacuum expectation values \( \langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{i \omega} \rangle \) in the sine-Gordon model and \( \langle \mathcal{L}_{-2} \mathcal{L}_{-2} \Phi_{1,3} \rangle \) in \( \Phi_{1,3} \) perturbed minimal CFT. We discuss applications of these results to short-distance expansions of two-point correlation functions. © 1999 Elsevier Science B.V.

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1. Introduction

One-point vacuum expectation values (VEV) of local fields are important characteristics of the quantum field theory (QFT) vacuum. Operator product expansions (OPE) \cite{1} give rise to short-distance expansions for multipoint correlation functions which involve the one-point VEV as the basic ingredients \cite{2,3}. At the same time the one-point VEV are non-perturbative objects and no systematic techniques for their evaluation is known. Some results for these quantities are available from numerical analyses (see, e.g., Ref. \cite{4} for numerical results in 2D QFT). Recently some progress has been made in the evaluation of the one-point VEV in 2D integrable QFT \cite{5-7}. In these
papers the one-point VEV of the primary fields in some integrable QFT, including the sine-Gordon model and $\Phi_{1,3}$ perturbed minimal CFT, were found exactly. On the other hand, complete characterization of the correlation functions requires the knowledge of the VEV of all local fields, including the descendant operators. In the present paper we address the problem of calculating the one-point VEV of descendant fields. It was shown in [6,7] that the VEV of the primary fields in the sine-Gordon model (and in similar integrable QFT) satisfy the remarkable “reflection relation” which involves the “reflection S-matrix” of Liouville CFT [8], and their one-point VEV can be obtained as appropriate solutions to these relations. We will show here how this approach can be extended to the descendant fields and explicitly evaluate the VEV of the simplest non-trivial descendants in the sine-Gordon model and in $\Phi_{1,3}$ perturbed minimal models.

Let us introduce basic notations and state the main results of this work. The sine-Gordon model is defined by the Euclidean action

$$A_{SG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial \varphi)^2 - 2\mu \cos(\beta \varphi) \right\},$$

(1.1)

where $\mu$ and $\beta$ are parameters, $0 < \beta^2 < 1$. The simplest local fields in this QFT are the exponentials $e^{ia\varphi}$. Exact VEV $\langle e^{ia\varphi} \rangle_{SG}$ of these fields are found in [5]. Here we will consider more general local fields of the form

$$(\partial^{m_1} \varphi)(\partial^{m_2} \varphi) \cdots (\partial^{m_k} \varphi)(\bar{\partial}^{m_1} \varphi)(\bar{\partial}^{m_2} \varphi) \cdots (\bar{\partial}^{m_k} \varphi)e^{ia\varphi}, \quad \text{Re} \alpha < \frac{1}{2\beta},$$

(1.2)

where $\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$ and $z, \bar{z}$ are complex coordinates, $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$. Precise definition of these fields in (1.1) requires specification of their renormalizations. We adopt the scheme in which the renormalized fields (1.2) have definite scale dimensions (see, e.g., Refs. [9,3]). Some details are presented in Section 2. In Section 3 we generalize the “reflection relations” of Refs. [6,7] to the fields (1.2) and use these relations to obtain the one-point VEV of the simplest non-trivial field of this kind,

$$\langle (\partial \varphi)^2(\bar{\partial} \varphi)^2 e^{ia\varphi} \rangle_{SG} = -\langle e^{ia\varphi} \rangle_{SG} \left[ M \frac{\sqrt{\pi} \Gamma\left(\frac{3}{2} + \frac{\beta^2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \right]^4 \times \frac{\Gamma\left(-\frac{\beta^2}{\beta^2} + \frac{\xi}{2}\right) \Gamma\left(\frac{\alpha}{\beta^2} + \frac{\xi}{2}\right) \Gamma\left(-\frac{1}{2} - \frac{\alpha}{\beta^2} - \frac{\xi}{2}\right) \Gamma\left(-\frac{1}{2} + \frac{\alpha}{\beta^2} - \frac{\xi}{2}\right)}{\Gamma\left(1 + \frac{\alpha}{\beta^2} - \frac{\xi}{2}\right) \Gamma\left(1 - \frac{\alpha}{\beta^2} - \frac{\xi}{2}\right) \Gamma\left(\frac{3}{2} + \frac{\alpha}{\beta^2} + \frac{\xi}{2}\right) \Gamma\left(\frac{3}{2} - \frac{\alpha}{\beta^2} + \frac{\xi}{2}\right)},$$

(1.3)

where

$$\xi = \frac{\beta^2}{1 - \beta^2}$$

(1.4)

and $M$ is the sine-Gordon soliton mass which relates to the parameter $\mu$ in (1.1) as [10]

$$\mu = \frac{\Gamma(\beta^2)}{\pi \Gamma(1 - \beta^2)} \left[ M \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{\beta^2}{2}\right)}{2 \Gamma\left(\frac{\beta^2}{2}\right)} \right]^{2-2\beta^2}. \quad (1.5)$$
Let us quote here the simpler form the expression (1.3) assumes in the case $\alpha = 0$,\footnote{It is interesting to note that Eq. (1.6) implies a remarkably simple relation between the VEV of different fields associated with the sine-Gordon energy-momentum tensor $T_{\mu\nu}$. Denoting as usual $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$ and $\theta = T_{z\bar{z}} = \frac{i}{2} T^\rho_\rho$ the irreducible spin components of $T_{\mu\nu}$ and using the known result for $\langle \theta^2 \rangle_{SG}$ \cite{11} we have $\langle T \rangle_{SG} = -\langle \theta^2 \rangle_{SG}$.}

$$
\langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 \rangle_{SG} = -\pi^2 M^4 \tan^2(\pi \xi/2). \tag{1.6}
$$

The sine-Gordon QFT is closely related to the minimal CFT \cite{12} perturbed by the operator $\Phi_{1,3}$, i.e. the QFT defined by the action

$$
\mathcal{M}_{p/p'} + \lambda \int d^2x \Phi_{1,3}(x), \tag{1.7}
$$

where $\mathcal{M}_{p/p'}$ stands for the formal action of the minimal model. Here we consider the massive case $\lambda > 0$. As is well known, this CFT can be obtained from (1.1) with

$$
\xi = \frac{p'}{p' - p}
$$

by the quantum group restriction \cite{13,14}. This relation was used in \cite{5,7} to obtain the VEV of all primary fields $\Phi_{l,k}$ in the QFT (1.7). In a similar fashion, the fields (1.2) in (1.1) are related to the descendant fields in (1.7). In particular, the result (1.3) is sufficient to derive the expectation values of the descendant fields $L_{-2} L_{-2} \Phi_{l,k}$, namely

$$
\frac{\langle 0_s | L_{-2} L_{-2} \Phi_{l,k} | 0_s \rangle}{\langle 0_s | \Phi_{l,k} | 0_s \rangle} = - \left[ M \sqrt{\pi} \Gamma(\frac{1}{2} + \frac{\xi}{2}) \right]^4 \mathcal{W}(\xi + 1) - \xi k, \tag{1.8}
$$

$$
\mathcal{W}(\eta) = \xi^{-2} (1 + \xi)^{-2} \frac{\Gamma \left( \frac{1}{2} + \frac{\xi}{2} \right) \Gamma \left( \frac{\eta - \xi}{2} \right) \Gamma \left( \frac{1 + \xi - \eta}{2} \right) \Gamma \left( \frac{1 + \xi + \eta}{2} \right) \Gamma \left( \frac{1 + \xi + \eta}{2} \right)}{\Gamma \left( \frac{1 + \eta - \xi}{2} \right) \Gamma \left( \frac{1 + \xi - \eta}{2} \right) \Gamma \left( \frac{1 + \xi + \eta}{2} \right) \Gamma \left( \frac{1 + \xi + \eta}{2} \right)}.
$$

Here $| 0_s \rangle$, $s = 1, 2, \ldots, p - 1$ is any one of $p - 1$ degenerate ground states of the CFT (1.7), and $M$ is the mass of its fundamental kink. This mass is related to $\lambda$ in (1.7) as \cite{10}

$$
\lambda^2 = \frac{(1 + \xi)^4}{(1 - \xi)^2 (1 - 2\xi)^2 \pi^2} \frac{\Gamma \left( \frac{\xi}{1 + \xi} \right) \Gamma \left( \frac{\xi}{1 + \xi} \right) \Gamma \left( \frac{3\xi}{1 + \xi} \right) \left[ M \sqrt{\pi} \Gamma \left( \frac{1}{2} + \frac{\xi}{2} \right) \right]^8}{\Gamma \left( \frac{1}{1 + \xi} \right) \Gamma \left( \frac{1 - 2\xi}{1 + \xi} \right) \left[ 2 \Gamma \left( \frac{\xi}{2} \right) \right]^8 \left(1 + \xi\right)}. \tag{1.9}
$$

In fact, the “reflection relations” admit certain ambiguity to their solution, akin to the “CDD ambiguity” in the factorizable S-matrix theory (see, e.g., Ref. [15]). To some extent the ambiguity can be narrowed by taking into account the “resonance conditions” (see Section 2). In (1.3) we have fixed this ambiguity by choosing the “minimal solution” – the simplest solution compatible with the resonance conditions (choosing the minimal solution is a common practice in the S-matrix theory). This choice is confirmed in Sections 4 and 5, where (1.3) is checked against results obtained in (1.1) using semiclassical approximation (Section 4) and ordinary Feynmann perturbation theory (Section 5). Moreover, the special case (1.6) can be obtained directly from exact lattice theory of the XYZ model, as we show in Section 6. Finally, in Section 7
we use Eq. (1.8) (more precisely, its particular case $l = k = 0$) to extend by one more order the short-distance expansion of the two-point correlation function of the scaling Lee–Yang Model [3].

2. Descendant fields and operator product expansions

The sine-Gordon model (1.1) can be regarded as Gaussian CFT

$$\mathcal{A}_{\text{Gauss}} = \int d^2 x \frac{1}{16\pi} (\partial_x \phi)^2$$

perturbed by the relevant operator $2 \cos(\beta \phi)$. Let $\mathcal{F}_{\text{Gauss}}$ be the space of local fields in the CFT (2.1). This space is spanned by the fields (1.2). In the free-field theory (2.1) these composite fields are defined through usual Wick ordering with

$$\langle \phi(z, \bar{z}) \phi(0, 0) \rangle_{\text{Gauss}} = -2 \log(z \bar{z}).$$

With this definition the field (1.2) has the dimensions $(\Delta, \bar{\Delta}) = (\alpha^2 + l, \alpha^2 + \bar{l})$, where the integer “levels” $l, \bar{l}$ are the sums $l = \sum_{i=1}^{N} n_i, \bar{l} = \sum_{j=1}^{K} m_j$. The sum

$$D = \Delta + \bar{\Delta} = 2\alpha^2 + l + \bar{l}$$

is the scale dimension of the field (1.2) while the difference $S = l - \bar{l}$ coincides with its spin. Note that some linear combinations of the fields (1.2) are total derivatives of other local fields, for example,

$$-\alpha^2 (\partial \phi)(\bar{\partial} \phi) e^{i\alpha \phi} = \partial \bar{\partial} e^{i\alpha \phi},$$

$$\partial^2 \phi \partial^2 \phi e^{i\alpha \phi} + i\alpha (\partial \phi)^2 (\bar{\partial} \phi)^2 e^{i\alpha \phi} = \partial \bar{\partial} (\partial \phi e^{i\alpha \phi}),$$

$$\partial^2 \phi \partial^2 \phi e^{i\alpha \phi} + i\alpha (\partial \phi)^2 (\bar{\partial} \phi)^2 e^{i\alpha \phi} = \partial \bar{\partial} (\partial \phi e^{i\alpha \phi}).$$

These elementary relations follow from the equation of motion of (2.1),

$$\partial \bar{\partial} \phi = 0.$$
counterterms) has definite scale dimension, which then coincides with (2.2). We say that the field $O_i$ has $n$th order resonance with the field $O_j$ if the dimensions of these fields satisfy the equation

$$D_i = D_j + 2n(1 - \beta^2)$$  \hspace{1cm} (2.6)

with some positive integer $n$. If this resonance condition is satisfied an obvious ambiguity

$$O_i \rightarrow O_i + \text{const.} \mu^n O_j$$  \hspace{1cm} (2.7)

in defining the renormalized field $O_i$ with the scale dimension $D_i$ typically results in the logarithmic scaling of $O_i$.

We define the field (1.2) in the perturbed theory (1.1) as the renormalized field satisfying the following conditions: (i) (1.2) has definite scale dimension (2.2) and (ii) the short-distance limit of its correlations coincides with the correlations of the corresponding field (1.2) in CFT (2.1). As explained above, in non-resonant cases this definition is unambiguous. In resonant cases it is not complete and one has to impose additional conditions to fix the ambiguity (2.7). For a given field (1.2) the resonances (2.6) occur at isolated values of $\alpha$. In this paper we are interested in generic values of this parameter and so we will not elaborate any specific convention concerning resonant cases. It suffices to note that if the matrix elements of (1.2) are viewed as the functions of $\alpha$ the resonances show up as the poles in this variable.

It is important to note that under this definition of the fields (1.2) in (1.1) these fields satisfy exactly the same relations (2.3), (2.4) (as well as similar higher-level relations) as the fields in (2.1). In this sense the symbol $\phi$ in (1.2) is the subject to the “free” equation of motion (2.5) rather than to the full equation of motion of the sine-Gordon theory. This is not a contradiction because it is rather the unrenormalized fields in (1.1) that satisfy the full equation of motion; the renormalized ones differ from those by counterterms. In this paper we are interested in the one-point VEV of the fields (1.2) in the QFT (1.1). Let us mention here some elementary properties of these VEV. First, only the fields of zero spin, i.e. with $l = 1$, have non-vanishing one-point VEV. Next, the one-point VEV of the fields which are total derivatives vanish. According to (2.3), (2.4) the only spinless field of the level $l = 1$ has vanishing VEV.

$$\langle (\partial \phi)(\bar{\partial} \phi)e^{i\alpha \phi} \rangle_{SG} = 0.$$  \hspace{1cm} (2.8)

There are four linearly independent spinless fields on the level $l = 2$. However, the following relations among their VEV are simple consequences of (2.4):

$$\langle (\partial^2 \phi)(\bar{\partial}^2 \phi)e^{i\alpha \phi} \rangle_{SG} = -i\alpha \langle (\partial \phi)^2(\bar{\partial} \phi)e^{i\alpha \phi} \rangle_{SG}$$

$$= -i\alpha \langle (\partial^2 \phi)^2 e^{i\alpha \phi} \rangle_{SG} = -\alpha^2 \langle (\partial \phi)^2(\bar{\partial} \phi)^2 e^{i\alpha \phi} \rangle_{SG}. \hspace{1cm} (2.9)$$

These relations allow one to express all $l = 2$ VEV through the VEV

$$\langle (\partial \phi)^2(\bar{\partial} \phi)^2 e^{i\alpha \phi} \rangle_{SG}. \hspace{1cm} (2.10)$$
Starting from the level $I = 3$ there are additional "kinematic" relations among the VEV following from the existence of higher local integrals of motion in the QFT \((1.1)\) [16], but we will not discuss them here. Instead we will concentrate attention on the VEV \((2.10)\).

Let us make a simple remark here concerning the properties of the VEV \((2.10)\) as the function of $\alpha$. It is easy to check that the field \(e^{i(\alpha+2\beta)\varphi}\) has a second-order resonance with the field \(e^{i(\alpha+2\beta)\varphi}\) at $\alpha = -\beta/2$. Similarly, at $\alpha = \beta/2$ it has second-order resonance with $e^{i(\alpha-2\beta)\varphi}$. Therefore, the VEV \((2.10)\) is expected to have poles at $\alpha = \pm\beta/2$. As we will argue below the residues at these poles can be expressed through the VEV of the primary fields responsible for the resonances.

To explain this point let us consider a product of two primary fields $e^{i\alpha_1\varphi}(x)e^{i\alpha_2\varphi}(y)$ in \((1.1)\). The corresponding OPE has the form
\[
e^{i\alpha_1\varphi}(x)e^{i\alpha_2\varphi}(y) = \sum_{n=-\infty}^{\infty} \left\{ C_{\alpha_1\alpha_2}^{n,0}(r) e^{i(n+\beta)\varphi}(y) + \ldots \right\},
\]
where $\alpha = \alpha_1 + \alpha_2$, $r = |x - y|$, and the dots in each term stand for contributions of the descendants \((1.2)\) of the field $e^{i(\alpha+n\beta)\varphi}(y)$. The coefficient functions $C$ are in principle computable within the conformal perturbation theory (CPT) [3] (see also Ref. [17]). The CPT suggests for them the following form:
\[
C_{\alpha_1\alpha_2}^{n,0}(r) = \mu^n |r|^{4\alpha_1\alpha_2+4n\beta(\alpha_1+\alpha_2)+2|n(1-\beta^2)+2n\beta^2 f_{\alpha_1\alpha_2}^{n,0}\mu^2 r^{4-4\beta^2}},
\]
where the functions $f$ in \((2.12)\) admit power series expansions, i.e.
\[
f_{\alpha_1\alpha_2}^{n,0}(t) = \sum_{k=0}^{\infty} f_{\alpha_1\alpha_2}^{n,0}(\alpha_1, \alpha_2) t^k.
\]
The CPT gives the coefficients in \((2.13)\) in terms of certain $2|n|+2k$-fold Coulomb-type integrals. Note that the leading terms $f_{\alpha_1\alpha_2}^{n,0}(\alpha_1, \alpha_2)$ in the series \((2.13)\) are expressed through the integrals
\[
j_n(a, b, \rho) = \frac{1}{n!} \int \prod_{k=1}^{n} d^2 x_k \prod_{k=1}^{n} |x_k|^{4a} |1 - x_k|^{4b} \prod_{k<p}^{n} |x_k - x_p|^{4\rho},
\]
\[(2.14)\]
namely
\[
f_{0,0}^{0}(\alpha_1, \alpha_2) = 1,
f_{0,0}^{0}(\alpha_1, \alpha_2) = j_n(\alpha_1\beta, \alpha_2\beta, \beta^2) \quad \text{for} \quad n > 0,
f_{0,0}^{0}(\alpha_1, \alpha_2) = j_n(-\alpha_1\beta, -\alpha_2\beta, \beta^2) \quad \text{for} \quad n < 0.
\]
\[(2.15)\]
The integrals \((2.14)\) are evaluated explicitly [18],
\[
j_n(a, b, \rho) = \pi^n (\gamma(\rho))^{-n} \prod_{k=1}^{n} \gamma(k\rho) \prod_{k=0}^{n-1} \gamma(1 + 2a + k\rho) \gamma(1 + 2b + k\rho) \times \gamma(-1 - 2a - 2b - (n - 1 + k)\rho).
\]
\[(2.16)\]
Here and below the notation \( y(t) = \Gamma(t)/\Gamma(1-t) \) is used. Let us quote also the expression for the first subleading term in the expansion (2.12) of the function \( C_{\alpha_1, \alpha_2}^{0,0} \),

\[
f_1^{0,0}(\alpha_1, \alpha_2) = J(\alpha_1 \beta, \alpha_2 \beta, \beta^2),
\]

(2.17)

where

\[
J(a, b, \rho) = \int d^2x \, d^2y |x|^{4a} |y|^{4b} |1-x|^{4a} |1-y|^{4b} |x-y|^{-4\rho}.
\]

(2.18)

This integral can be expressed through the generalized hypergeometric function \( _3F_2 \) at unity [19] (see also Ref. [20]). The coefficient functions standing in front of the descendant field in (2.11) admit similar CPT expansions.

There are reasons to believe that the series (2.13) (and similar CPT series for the coefficient functions corresponding to the descendant fields in (2.11)) converge for all complex \( t \). But independently of the convergence these series can be used to generate asymptotic short-distance expansion for the two-point correlation function

\[
G_{\alpha_1, \alpha_2}(r) = \langle e^{i\alpha \varphi(x)} e^{i\alpha \varphi(y)} \rangle_{SG}, \quad r = |x-y|,
\]

(2.19)

provided the one-point VEV of the exponential fields in the r.h.s. of (2.11), and also the one-point VEV of their descendants (1.2), are known. For the exponential fields the one-point VEV

\[
G_{\alpha} = \langle e^{i\alpha \varphi} \rangle_{SG}
\]

closed analytic expression exists [5]. According to our discussion above the first non-zero contribution due to the descendants comes from the fields \( (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{i(\alpha+n\beta) \varphi} \), namely

\[
G_{\alpha_1, \alpha_2}(r) = \sum_{n=-\infty}^{\infty} \left\{ C_{\alpha_1, \alpha_2}^{n,0}(r) \langle e^{i(\alpha+n\beta) \varphi} \rangle_{SG} + C_{\alpha_1, \alpha_2}^{n,2}(r) \langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{i(\alpha+n\beta) \varphi} \rangle_{SG} + \ldots \right\},
\]

(2.21)

where \( \alpha = \alpha_1 + \alpha_2 \) and the omitted terms contain the descendants of the levels \( l = \tilde{l} = 4 \) or higher. The coefficient functions \( C_{\alpha_1, \alpha_2}^{n,2}(r) \) admit CPT expansions similar to (2.12), (2.13). In particular,

\[
C_{\alpha_1, \alpha_2}^{0,2}(r) = -\frac{(\alpha_1 \alpha_2)^2}{4} r^4 \left( 1 + O(\mu^2 r^{4-4\beta^2}) \right).
\]

(2.22)

Combining all these expressions one can write down the \( r \to 0 \) expansion

\[
G_{\alpha_1, \alpha_2}(r) = G_{\alpha_1 + \alpha_2} r^{4\alpha_1 \alpha_2} \left\{ 1 + J(\alpha_1 \beta, \alpha_2 \beta, \beta^2) \mu^2 r^{4-4\beta^2} \right. \\
- \frac{(\alpha_1 \alpha_2)^2}{4} \mathcal{H}(\alpha_1 + \alpha_2) r^4 + O(\mu^4 r^{8-8\beta^2}) \\
+ \sum_{n=1}^{\infty} G_{\alpha_1 + \alpha_2 + n\beta} J_n(\alpha_1 \beta, \alpha_2 \beta, \beta^2) \left\{ 1 + O(\mu^2 r^{4-4\beta^2}) \right\}
\]

(2.22)
\[ X \{ I - O (~) ~ 2 \} ^{2} \sum_{n=1}^{\infty} \mathcal{G} \mathcal{A} - \eta \beta j_{n} (-\alpha_{1} \beta, -\alpha_{2} \beta, \beta^{2}) \]

\[ \times \left\{ 1 + O(\mu^{2}r^{4-4\beta^{2}}) \right\} \mathcal{H}^{n} \mathcal{A} \mathcal{A} - 4n \beta (\alpha_{1} + \alpha_{2}) + 2n(1-\beta^{2}) + 2n^{2} \beta^{2}, \]  

(2.23)

where \( \mathcal{H}(\alpha) \) stands for the ratio

\[ \mathcal{H}(\alpha) = \frac{\langle (\partial \varphi)^{2} (\bar{\varphi})^{2} e^{i\alpha \varphi} \rangle_{SG}}{\langle e^{i\alpha \varphi} \rangle_{SG}}. \]  

(2.24)

Note that at \( \alpha = -\beta/2 \) the leading contribution of the field \( e^{i(\alpha+2\beta)\varphi} \) has the same power low in \( r \) as the contribution of the descendant \( (\partial \varphi)^{2} (\bar{\varphi})^{2} e^{i\alpha \varphi} \). This is exactly the second-order resonance we have mentioned above. The contribution comes with the coefficients \( j_{2} \) which exhibit the pole at this value of \( \alpha \) as is seen from (2.16). The VEV (2.10) also has a resonance pole at this point, and these two pole terms must compensate. This requirement leads to the relation

\[ \text{res}_{\alpha=-\beta/2} \mathcal{H}(\alpha) = \left( \frac{\pi \mu}{\gamma (\beta^{2})} \right)^{2 + 2\xi} \frac{4}{\beta} (1 + \xi)^{3} \gamma \left( -\frac{1}{2} - \xi \right) \gamma(\xi). \]  

(2.25)

The compensation of the poles at \( \alpha = -\beta/2 \) results in the logarithmic term of \( r^{4\alpha_{1}\alpha_{2}+4} \log(r) \) in the short-distance expansion of (2.19) with \( \alpha_{1} + \alpha_{2} = -\beta/2 \). The relation (2.25) will be used in the next section to fix the normalization of the VEV (2.10).

### 3. Reflection relations

In Ref. [7] the "reflection property" of the Liouville CFT was used to derive the one-point VEV of the exponential fields \( e^{i\alpha \varphi} \) in sine-Gordon model. Let us first briefly recall the arguments of Ref. [7], and then show how these arguments can be extended to the case of the descendant fields (1.2).

The sine-Gordon model (1.1) is closely related to the sinh-Gordon model

\[ \mathcal{A}_{\text{shG}} = \int d^{2}x \left\{ \frac{1}{16\pi} (\partial_{\nu} \varphi)^{2} + 2\mu \cosh(b \varphi) \right\}. \]  

(3.1)

In particular, the one-point VEV \( \langle e^{i\alpha \varphi} \rangle_{SG} \) and \( \langle e^{i\alpha \varphi} \rangle_{\text{shG}} \) in the two models are related through the substitution

\[ b = i\beta, \quad a = i\alpha. \]  

(3.2)

This relation also holds for the one-point VEV of the descendant fields (1.2). In turn, the sinh-Gordon model can be regarded as the Liouville CFT,
\[ \mathcal{A}_L = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\mu \phi)^2 + \mu e^{b\phi} \right\}, \]  

perturbed by the operator \( e^{-b\phi} \). As is known [8], the correlation functions of the fields \( e^{a\phi} \) in the Liouville theory exhibit the following "reflection property":

\[ \langle e^{a\phi}(x) \ldots \rangle_L = R(a) \langle e^{(Q-a)\phi}(x) \ldots \rangle_L, \]  

where

\[ Q = b^{-1} + b \]  

and the coefficient function

\[ R(a) = \frac{\Gamma\left( \frac{1}{2} - \frac{a}{b} \right) \Gamma\left( \frac{1}{2} + \frac{2a}{b} \right)}{\Gamma(2 + b^2 - 2ab) \Gamma(2 + b^2 - 2ab)} \]  

is essentially the vacuum reflection amplitude of the Liouville CFT. The relation (3.4) suggest the following "reflection relation" for the one-point VEV of (3.1),

\[ \langle e^{a\phi} \rangle_{shG} = R(a) \langle e^{(Q-a)\phi} \rangle_{shG}. \]  

Combining this relation with the obvious symmetry property

\[ \langle e^{a\phi} \rangle_{shG} = \langle e^{-a\phi} \rangle_{shG} \]  

and certain assumptions about analytic properties of \( \langle e^{a\phi} \rangle_{shG} \) [7] one can derive VEV for exponential field.

Conformal symmetry of the Liouville theory (3.3) is generated by the energy-momentum tensor

\[ T_L(z) = -\frac{1}{4}(\partial \phi)^2 + \frac{Q}{2} \partial^2 \phi, \]  
\[ \bar{T}_L(\bar{z}) = -\frac{1}{4}(\bar{\partial} \phi)^2 + \frac{\bar{Q}}{2} \bar{\partial}^2 \phi. \]  

The exponentials \( e^{a\phi} \) are primary fields with respect to the Virasoro algebra generated by (3.9). Let us introduce the notation

\[ L_{[-n]} \bar{L}_{[m]} e^{a\phi} \equiv L_{-n_1} L_{-n_2} \ldots L_{-n_N} \bar{L}_{-m_1} \bar{L}_{-m_2} \ldots \bar{L}_{-m_K} e^{a\phi} \]  

for the corresponding descendant fields. The symbols \([n]\) and \([m]\) here stand for arbitrary strings \([-n_1, -n_2, \ldots, -n_N], \ [-m_1, -m_2, \ldots, -m_K]\). In (3.10) \( L_n, \bar{L}_n \) are standard Virasoro generators associated with (3.9). It is possible to show that the reflection property extends to all these descendants, namely

\[ \langle L_{[-n]} \bar{L}_{[m]} e^{a\phi}(x) \ldots \rangle_L = R(a) \langle L_{[n]} \bar{L}_{[m]} e^{(Q-a)\phi}(x) \ldots \rangle_L. \]  

The arguments identical to those in [7] suggest then that the "reflection relation" (3.7) generalizes to the descendant fields (3.10) in a straightforward way,
The generalization of the symmetry relation (3.8) is less straightforward. The relation (3.8) is a simple consequence of the symmetry \( \varphi \rightarrow -\varphi \) of the action (3.1). However, while the action (3.1) is invariant with respect to this transformation, the components (3.9) of the modified energy–momentum tensor, and hence the corresponding Virasoro generators \( L_n, L_m \), are not. In this respect the basis

\[
(\partial^{m_1} \varphi)(\partial^{m_2} \varphi) \ldots (\partial^{m_k} \varphi)(\bar{\partial}^{m_1} \varphi)(\bar{\partial}^{m_2} \varphi) \ldots (\bar{\partial}^{m_k} \varphi) e^{a \varphi}
\]

in the space of the descendants is more convenient as the fields (3.13) transform under this reflection in an obvious way. The fields (3.10) can be written down as the linear combinations of the fields (3.13) of the same levels \( l, \bar{l} \) and vice versa.\(^3\) Finding this relation for given levels requires solving a finite algebraic problem, as explained in [8]. Here we will only need the relation [8]

\[
L_{-2} e^{a \varphi} = \frac{-1}{4}(\partial \varphi)^2 + \left( \frac{Q}{2} + a \right) \partial^2 \varphi \left[ \frac{-1}{4}(\bar{\partial} \varphi)^2 + \left( \frac{Q}{2} + a \right) \bar{\partial}^2 \varphi \right] e^{a \varphi}.
\]

Consider the one-point VEV of the field (3.14),

\[
\langle L_{-2} \bar{L}_{-2} e^{a \varphi} \rangle_{\text{shG}} = \frac{1}{16} \left( 1 + 2a(Q + 2a) \right)^2 \langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{a \varphi} \rangle_{\text{shG}},
\]

where the relations analogous to (2.9) were used to simplify the r.h.s. Then it follows from (3.12) that

\[
\left( 1 + 2a(Q + 2a) \right)^2 \langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{a \varphi} \rangle_{\text{shG}} = R(a) \left( 1 + 2(Q - a)(3Q - 2a) \right)^2 \langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{(Q - a) \varphi} \rangle_{\text{shG}}.
\]

We find that the function

\[
H(a) = \frac{\langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 e^{a \varphi} \rangle_{\text{shG}}}{\langle e^{a \varphi} \rangle_{\text{shG}}}
\]

satisfies the functional equations

\[
H(a) = \left[ \frac{(2b + 3/b - 2a)(3b + 2/b - 2a)}{b + 2a(1/b + 2a)} \right]^2 H(Q - a),
\]

\[
H(a) = H(-a),
\]

where the second equation follows from the obvious symmetry of (3.17). Note that Eq. (3.18) remains unchanged if one makes the substitution

\(^3\) Of course, if the Virasoro module with the primary field \( e^{a \varphi} \) has a null vector at the level \( l \) the relation between (3.10) and (3.13) becomes singular. Below we consider generic case of \( a \) and ignore this subtlety.
This is in agreement with well-known "duality" symmetry of the sinh-Gordon model (3.1), which in particular implies that all VEV of the fields (3.13) must be invariant with respect to the transformation (3.19).

Obviously, Eqs. (3.18) determine the function $H(a)$ only up to a factor $F(a)$ which is an even periodic function,

$$F(a) = F(-a), \quad F(a) = F(a + Q).$$

The solution we are interested in must have the poles at $a = \pm b/2$ corresponding to the second-order resonances discussed in the previous section. Also, the function $H(a)$ must respect the symmetry (3.19). Strictly speaking, this information is not sufficient to fix the ambiguity (3.20). Nevertheless, there is a "minimal" solution which satisfies the above requirements,

$$H(a) = \frac{m\Gamma(b/2Q)\Gamma(1/2bQ)}{8Q^2\sqrt{\pi}} \gamma\left(\frac{a}{Q} - \frac{b}{2Q}\right) \gamma\left(-\frac{a}{Q} - \frac{b}{2Q}\right) \times \gamma\left(\frac{a}{Q} - \frac{1}{2bQ}\right) \gamma\left(-\frac{a}{Q} - \frac{1}{2bQ}\right),$$

where $\gamma(t) = \Gamma(t)/\Gamma(1-t)$ and $m$ is the mass of the sinh-Gordon particle. The residue condition analogous to (2.25) is used to fix the overall normalization of (3.21). We conjecture that this minimal solution gives exact ratio (3.17) in the sinh-Gordon model. The VEV (2.10) is then obtained by the substitution (3.2), which yields (1.3). In the subsequent sections we give some evidence in support of this conjecture.

4. Comparison to semiclassical results

The result (1.3) can be checked against certain semiclassical calculations in (1.1). Consider the two-point correlation function (2.19) with

$$\alpha_1 = \omega \beta, \quad \alpha_2 = \sigma / \beta,$$

where both $\sigma, \omega \sim 1$, in the limit $\beta \to 0$. In this limit the functional integral defining (2.19) is dominated by the saddle-point configuration $\varphi_{cl}(x) = (2i/\beta)\phi(t)$, $t = m|x-y|$, where $\phi(t)$ is a solution to the Painlevé III equation

$$\partial_t^2 \phi + t^{-1} \partial_t \phi = \frac{1}{2} \sinh(2\phi)$$

regular at $t > 0$ and satisfying the asymptotic conditions

$$\phi(t) = 2\sigma \log(8/t) - \log \left[ \gamma\left(\frac{1}{2} - \sigma\right) \right] + O\left(t^{2\pm 4\sigma}\right) \quad \text{as} \quad t \to 0,$$

$$\phi(t) \to \frac{2\sin(\pi \sigma)}{\pi} K_0(t) \quad \text{as} \quad t \to +\infty,$$
where $K_0(t)$ is the MacDonald function and again $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$. Therefore, the correlation function under consideration can be written as

$$\left. \frac{\langle e^{i\omega\beta\varphi(x)} e^{i\beta\varphi(y)} \rangle_{SG}}{\langle e^{i\beta\varphi} \rangle_{SG} \langle e^{i\beta\varphi} \rangle_{SG}} \right|_{\beta^2 \to 0} = (e^{2\phi(t)})^{-\omega}. \quad (4.4)$$

As is known [21], this solution to the Painlevé III equation admits a double-series expansion

$$\sum_{m,n=0}^{\infty} (m + n + 2\sigma(m - n))^2 B_{m,n} \left( \frac{t}{8} \right)^{2(m+n-1+2\sigma(m-n))}, \quad (4.5)$$

where the coefficients $B_{m,n}$ satisfy certain recursion relations (see Ref. [21] for details). Using these relations one can derive explicitly few first terms of the expansion (4.5),

$$e^{2\phi(t)} = \gamma^2 \left( \frac{1}{2} + \sigma \right) \left( \frac{1}{8} \right)^{-4\sigma} + \frac{8}{(1 - 2\sigma)^2} \gamma^4 \left( \frac{1}{2} + \sigma \right) \left( \frac{1}{8} \right)^{2 - 8\sigma}$$

$$- \frac{8}{(1 + 2\sigma)^2} \left( \frac{1}{8} \right)^2 + \frac{48}{(1 - 2\sigma)^4} \gamma^6 \left( \frac{1}{2} + \sigma \right) \left( \frac{1}{8} \right)^{4 - 12\sigma}$$

$$- \frac{64(1 - 2\sigma)}{(1 - 4\sigma^2)^2} \gamma^2 \left( \frac{1}{2} + \sigma \right) \left( \frac{1}{8} \right)^{4 - 4\sigma}$$

$$+ \frac{16}{(1 + 2\sigma)^4} \gamma^2 \left( \frac{1}{2} - \sigma \right) \left( \frac{1}{8} \right)^{4 + 4\sigma} + O(t^{6 - 16\sigma}, t^6). \quad (4.6)$$

This expansion is to be compared with the corresponding limiting case of the expansion (2.23). To make this comparison straightforward one can use the relation

$$\frac{G_{\sigma/\beta + \omega\beta + n\beta}}{G_{\sigma/\beta} G_{\omega\beta}} \bigg|_{\beta^2 \to 0} \to \left( \frac{m}{8} \right)^{4(\omega + n)\sigma} \left[ \gamma \left( \frac{1}{2} - \sigma \right) \right]^{2\omega + 2n}. \quad (4.7)$$

which is obtained from the explicit formula for the VEV (2.20) [5], and the following limiting expressions for the integrals (2.18) and (2.16):

$$J(\sigma, \omega \beta^2, \beta^2) \bigg|_{\beta^2 \to 0} \to -8\pi^2 \beta^4 \frac{\omega(\sigma + \omega)}{(1 - 4\sigma^2)^2},$$

$$j_n(\sigma, \omega \beta^2, \beta^2) \bigg|_{\beta^2 \to 0} \to \frac{\pi^n \beta^{2n}}{(1 + 2\sigma)^{2n}} \frac{\Gamma(2\omega + n)}{n! \Gamma(2\omega)}. \quad (4.8)$$

Also, assuming (1.3) valid, one has for the ratio (2.24)

$$\mathcal{H}(\sigma/\beta + \omega\beta) \bigg|_{\beta^2 \to 0} \to -\frac{m^2}{16\sigma^2(1 - 4\sigma^2)^2}. \quad (4.9)$$

Finally, $\mu|_{\beta^2 \to 0} \to m^2/16\pi \beta^2$, and with (4.7), (4.8), (4.9) the expansion (2.23) takes the form
\[ \frac{\langle e^{i\omega \beta \varphi(x)} e^{i\beta\varphi(y)} \rangle_{SG}}{\langle e^{i\omega \beta \varphi} \rangle_{SG} \langle e^{i\beta \varphi} \rangle_{SG}} \bigg|_{\beta^2 \to 0} = \left( \frac{t}{8} \right)^4 \omega \left( \frac{1}{2} - \sigma \right)^2 \left\{ 1 - \frac{t^4 \omega (2\sigma + \omega)}{64 (1 - 4\sigma^2)^2} \right\} + O(t^8) + \sum_{n=1}^{\infty} \frac{\Gamma(2\omega + n)}{n! \Gamma(2\omega)} \left[ 2\gamma \left( \frac{1}{2} - \sigma \right) \right]^{2n} \left( \frac{t}{8} \right)^{2n(1 + 2\sigma)} \left( 1 + O(t^4) \right) \\
+ \sum_{n=1}^{\infty} \frac{\Gamma(-2\omega + n)}{n! \Gamma(-2\omega)} \left[ 2\gamma \left( \frac{1}{2} + \sigma \right) \right]^{2n} \left( \frac{t}{8} \right)^{2n(1 - 2\sigma)} \left( 1 + O(t^4) \right) \right\}, \quad (4.10) \]

where again \( t = m |x - y| \). It is not difficult to see now that (4.6) and (4.10) are in exact agreement with (4.4). Thus the semiclassical relation (4.4) actually yields (4.9) and therefore supports our main result (1.3). It is interesting to notice that agreement between (4.10) and (4.5) further suggests the following explicit expressions for some of the coefficients \( B_{m,n} \) in (4.5):

\[
B_{0,n} = \frac{4^n}{n(1 - 2\sigma)^{2n}} \gamma^{2n} \left( \frac{1}{2} + \sigma \right), \quad n = 1, 2, \ldots, \\
B_{m,0} = B_{m+2,1} = 0, \quad m = 1, 2, \ldots, \quad (4.11)
\]

which are not immediately obvious from the recursion relations of Ref. [21].

5. Comparison to perturbation theory

It is easy to see that (1.3) admits power series expansion in \( \beta^2 \). For further references, let us quote two special cases. First, for \( \alpha = 0 \)

\[ \langle (\partial \varphi)^2 (\bar{\varphi} \varphi)^2 \rangle_{SG} = -\frac{\pi^2 m^4}{4 \sin^2(\pi \xi)} = -\frac{m^4}{4\beta^4} + \frac{m^2}{2\beta^2} + O(1). \quad (5.1) \]

Here we have chosen to express the VEV (1.6) through the mass \( m = 2M \sin(\pi \xi/2) \) of the lightest sine-Gordon breather. Second, if \( \alpha = \omega \beta \), where \( \omega \) is a constant,

\[ \langle (\partial \varphi)^2 (\bar{\varphi} \varphi)^2 e^{i\beta_0 \varphi} \rangle_{SG} = -\frac{m^4}{4\beta^4} \frac{1}{1 - 4\omega^2} + O \left( \frac{1}{\beta^2} \right). \quad (5.2) \]

These expansions can be compared with the results obtained directly from (1.1) by means of the ordinary Feynmann perturbation theory.

In developing the perturbation theory it is convenient to start from the action (1.1) in its "bare" form

\[ A_{SG} = \int d^2x \left\{ \frac{1}{8\pi} \left( \frac{1}{2} (\partial \varphi)^2 - \frac{m^2}{\beta^2} [\cos(\beta \varphi)]_B \right) \right\} \]

\[ = \int d^2x \left\{ \frac{1}{8\pi} \left( \frac{1}{2} (\partial \varphi)^2 + \frac{m^2}{2} [\varphi^2]_B - \frac{m_0^2 \beta^2}{4!} [\varphi^4]_B + \ldots \right) \right\}, \quad (5.3) \]

where the symbol \([\ldots]_B\) signifies that we are dealing with the bare fields (as opposed to the renormalized fields defined in Section 2), and it is assumed that the action (5.3)
is supplemented with some cutoff procedure, with the cutoff momentum $A$. The scale dimensions of the bare fields coincide with their naive values and hence the bare mass parameter $m_0$ has the dimension of mass. The relation between $m_0$ and the physical mass $m$ of the lightest breather particle of (1.1) can be found perturbatively, order by order in $\beta^2$. For instance, with the account of the leading mass correction diagram in Fig. 1, we have

$$m_0^2 = m^2 + m^2 \beta^2 L + O(\beta^4),$$

(5.4)

where

$$L = \frac{1}{\pi} \int \frac{d^2k}{k^2 + m^2} = \log \left( \frac{\Lambda^2}{m^2} \right) + C,$$

(5.5)

and $C$ is a constant whose exact value depends on the implementation of the cutoff procedure. In fact, to all orders in $\beta^2$ this relation has the form [10]

$$m_0^2 = m^2 e^{\beta^2 L h(\beta^2)}; \quad h(\beta^2) = 1 + \frac{\pi^2}{6} \beta^4 + O(\beta^6).$$

(5.6)

Similarly, the relation between the bare exponential fields $[e^{i\alpha\varphi}]_B$ and corresponding renormalized fields can be written as

$$e^{i\alpha\varphi} = \left( \frac{4}{3} \Lambda^2 e^{2\gamma + C} \right)^{\alpha^2} [e^{i\alpha\varphi}]_B,$$

(5.7)

where $\gamma$ is Euler's constant and the normalization of the field $e^{i\alpha\varphi}$ is fixed by the short-distance asymptotic condition

$$\langle e^{i\alpha\varphi}(x)e^{-i\alpha\varphi}(y) \rangle_{SG} \to |x - y|^{-4\alpha^2} \quad \text{as} \quad |x - y| \to 0.$$  

(5.8)

Now we are prepared to make some perturbative calculations of the one-point VEV. As the first example let us consider the field $(\varphi^2)(\bar{\varphi}^2)$. According to our discussion in Section 2 this renormalized field differs from the corresponding bare field by appropriate counterterms

$$(\varphi^2)(\bar{\varphi}^2) = \left[ (\varphi^2)(\bar{\varphi}^2) \right]_B + m_0^4 \left( A_0 + A_1 \left[ \cos(2\beta\varphi) \right]_B \right) + m_0^2 A_2 \partial \bar{\partial} \left[ \cos(\beta\varphi) \right]_B,$$

(5.9)

where $A_0, A_1, A_2$ are constants which can depend on $\beta$. In (5.9) the counterterms of the type $\Lambda^2 \left[ \partial\varphi\bar{\partial}\varphi \right]_B$ and $\Lambda^4$ which are needed to absorb the quadratic divergences in the

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4 The exact form of the function $h(\beta^2)$ can be found in Ref. [10].
matrix elements of \([ (\partial \varphi)^2 (\bar{\partial} \varphi)^2 ]_B \) are not written down. In the following calculations of these matrix elements we will systematically subtract all quadratic divergences; with this convention the "quadratic" counterterms can be ignored altogether. The counterterms explicitly shown in (5.9) are to compensate for the remaining logarithmic divergences. It is possible to see that this compensation can not be achieved with the coefficient \( A_2 \) being just constant; instead one has to set \( A_2 = A L + B \), where \( L \) is the logarithm (5.5).

The reason for this subtlety lays in the fact that the field \((\partial \varphi)^2 (\bar{\partial} \varphi)^2 \) always has a first-order resonance with the field \( \partial \bar{\varphi} \cos(\beta \varphi) \), which results in the logarithmic scaling of all its matrix elements which receive contributions from the above total derivative field. Fortunately, here we are interested only in the one-point VEV

\[
\langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 \rangle_{SG} = \langle [(\partial \varphi)^2 (\bar{\partial} \varphi)^2]_B \rangle_{SG} + A_0 m^4 e^{2\beta L} h^2(\beta^2) + A_1 m^4 e^{-2\beta L} h^2(\beta^2) g(2\beta),
\]

which gets no contribution from the last counterterm in (5.9) and hence is not sensitive to the above subtlety. In writing (5.10) we have used (5.6) to express \( m_0 \) through the physical mass and also used the notation

\[
\langle e^{i\omega \varphi} \rangle_B = e^{-\omega^2 L} g(\alpha).
\]

The explicit expression for \( g(\alpha) \) can be found in Ref. [5]; here we will only use the fact that for fixed \( \omega \)

\[
g(\omega \beta) = 1 + O(\beta^6).
\]

The first term in (5.10) can be calculated directly using Feynmann diagrams for (5.3). To the leading order in \( \beta^2 \) one obtains

\[
\langle [(\partial \varphi)^2 (\bar{\partial} \varphi)^2]_B \rangle_{SG} = \frac{1}{2} m^4 L^2 + O(\beta^2).
\]

(Let us remind that we subtract the quadratic divergences). In order to compensate for this \( L^2 \) divergence, the counterterm coefficients in (5.10) have to be chosen as follows:

\[
A_0 = -\frac{1}{8\beta^4} + O\left(\frac{1}{\beta^2}\right), \quad A_1 = -\frac{1}{8\beta^4} + O\left(\frac{1}{\beta^2}\right)
\]

and we obtain

\[
\langle (\partial \varphi)^2 (\bar{\partial} \varphi)^2 \rangle_{SG} = -\frac{m^4}{4\beta^4} + O\left(\frac{1}{\beta^2}\right).
\]

The calculation can be easily extended to the next order in \( \beta^2 \). The next perturbative contribution to the VEV (5.13) comes from the diagram in Fig. 2. It has the form

\[
m^4 \beta^2 (a_1 L^2 + a_2 L + a_3),
\]

where \( a_1, a_2, a_3 \) are numerical coefficients. It is not difficult to check that in order to find the next term in (5.1) one only needs to know the coefficient \( a_1 \) in front of
the leading logarithmic term in (5.16). This coefficient is evaluated directly from the diagram, $a_1 = 1$. Then compensation of this term requires the following terms in the $\beta^2$ expansion (5.14)

$$A_0 = -\frac{1}{8\beta^4} + \frac{1}{4\beta^2} + O(1), \quad A_1 = -\frac{1}{8\beta^4} + \frac{1}{4\beta^2} + O(1). \quad (5.17)$$

The finite terms remaining in (5.10) after the cancellation of the divergences yield exactly (5.1).

Next, let us apply the perturbation theory to more general VEV (2.10) with $\alpha \neq 0$. Again, the renormalized field is a combination of corresponding bare field and suitable counterterms,

$$\left( \frac{A^2 e^{2y+C}/4}{\langle \phi^2 \rangle} \right)^{-\alpha^2} \langle \bar{\partial} \bar{\phi} \rangle^2 \langle \bar{\phi} \rangle^2 e^{i\alpha \phi} = \left[ \langle \bar{\partial} \bar{\phi} \rangle^2 \langle \bar{\phi} \rangle^2 e^{i\alpha \phi} \right]_B + m_0^2 \partial \bar{\phi} \left( B_+ \left[ e^{i\alpha \phi} \right]_B + B_- \left[ e^{i\alpha \phi} \right]_B \right) + m_0^4 \left( A_+ \left[ e^{i\alpha \phi} \right]_B + A_0 \left[ e^{i\alpha \phi} \right]_B + A_- \left[ e^{i\alpha \phi} \right]_B \right). \quad (5.18)$$

The constants $A, B$ have to be determined from the requirement that the renormalized field $\langle \bar{\partial} \bar{\phi} \rangle^2 \langle \bar{\phi} \rangle^2 e^{i\alpha \phi}$ has definite scale dimension $4 + 2\alpha^2$. It is convenient to divide (5.18) by the VEV (5.11), and trade the parameter $m_0$ in favor of $m$,

$$\frac{\langle \bar{\partial} \bar{\phi} \rangle^2 \langle \bar{\phi} \rangle^2 e^{i\alpha \phi}}{\langle e^{i\alpha \phi} \rangle_{SG}} = \frac{\left[ \langle \bar{\partial} \bar{\phi} \rangle^2 \langle \bar{\phi} \rangle^2 e^{i\alpha \phi} \right]_B}{\langle e^{i\alpha \phi} \rangle_B} + m^2 \partial \bar{\phi} \left( B_+ e^{-2\alpha \beta L} \frac{g(\alpha + \beta)}{g(\alpha)} \left[ e^{i\alpha \phi} \right]_N \right) + B_- e^{-2\alpha \beta L} \frac{g(\alpha - \beta)}{g(\alpha)} \left[ e^{i\alpha \phi} \right]_N \right) + m^4 h^2(\beta^2) \left( A_+ e^{-(2\beta^2 + 4\alpha \beta)} L \frac{g(\alpha + 2\beta)}{g(\alpha)} \left[ e^{i\alpha \phi} \right]_N \right) + A_0 e^{2\beta^2} + A_- e^{-(2\beta^2 - 4\alpha \beta)} L \frac{g(\alpha - 2\beta)}{g(\alpha)} \left[ e^{i\phi} \right]_N \right), \quad (5.19)$$

where we used the notation
One can calculate perturbatively matrix elements of (5.19), adjusting the coefficients order by order in $\beta$ to ensure the cancellation of all $L$-dependent terms. Note that (5.20) contains no divergences and so all the $L$ dependence of the counterterm part in (5.19) is shown explicitly. It turns out that contrary to the case $\alpha = 0$ studying just the VEV of (5.19) is not enough to determine the coefficients $A_+, A_0, A_-$. We have considered the matrix elements of (5.19) between the vacuum and one- and two-particle states (involving the lightest breather) along with the VEV. The calculations are straightforward but rather bulky and we do not present them here. In the case $\alpha = \omega \beta$, $\omega \sim 1$ and in the leading order in $\beta^2$ the cancellation of $L$-dependent terms requires the following choice of coefficients:

\begin{align*}
B_+ &= \frac{1}{\beta^4} \frac{1}{2\omega(1 + \omega)^2} + O\left(\frac{1}{\beta^2}\right), \\
B_- &= -\frac{1}{\beta^4} \frac{1}{2\omega(1 - \omega)^2} + O\left(\frac{1}{\beta^2}\right),
\end{align*}

and

\begin{align*}
A_+ &= -\frac{1}{\beta^4} \frac{1}{16(1 + \omega)(1 + 2\omega)} + O\left(\frac{1}{\beta^2}\right), \\
A_- &= -\frac{1}{\beta^4} \frac{1}{16(1 - \omega)(1 - 2\omega)} + O\left(\frac{1}{\beta^2}\right), \\
A_0 &= -\frac{1}{\beta^4} \frac{1}{8(1 + \omega)(1 - \omega)} + O\left(\frac{1}{\beta^2}\right).
\end{align*}

(5.21)

With (5.21) the result for the VEV of this field identical to (5.2) immediately follows from (5.19).

6. Exact results from XYZ model

As is well known [22], the sine-Gordon QFT (1.1) can be obtained by taking an appropriate scaling limit of the XYZ spin chain described by the Hamiltonian

\begin{equation}
H_{\text{XYZ}} = -\frac{1}{2\epsilon} \sum_{s=1}^{N} (J_x \sigma_x \sigma_{s+1}^x + J_y \sigma_y \sigma_{s+1}^y + J_z \sigma_z \sigma_{s+1}^z - J),
\end{equation}

(6.1)

with $J_x \geq J_y \geq |J_z|$. In (6.1) we have introduced an auxiliary parameter $\epsilon$, which is interpreted as a lattice spacing. It is convenient to use Baxter’s elliptic parameterization [23] of the coefficients $J$ in (6.1),

\begin{align*}
J_x &= \frac{1 - \beta^2}{\pi} \left( \frac{\theta_4^2(\beta^2) \theta_1'(0)}{\theta_4(0) \theta_1(\beta^2)} + \frac{\theta_1(\beta^2) \theta_4'(0)}{\theta_4(0) \theta_4(\beta^2)} \right), \\
J_y &= \frac{1 - \beta^2}{\pi} \left( \frac{\theta_4^2(\beta^2) \theta_1'(0)}{\theta_4(0) \theta_1(\beta^2)} - \frac{\theta_1(0) \theta_4'(0)}{\theta_4(0) \theta_4(\beta^2)} \right),
\end{align*}

where $\theta_4(\beta^2)$ and $\theta_1(\beta^2)$ are elliptic functions.
\[ J_z = \frac{1 - \beta^2}{\pi} \left( \frac{\theta'_1(\beta^2)}{\theta_1(\beta^2)} - \frac{\theta'_4(\beta^2)}{\theta_4(\beta^2)} \right), \]
\[ J = -\frac{1 - \beta^2}{\pi} \left( \frac{\theta'_1(\beta^2)}{\theta_1(\beta^2)} + \frac{\theta'_4(\beta^2)}{\theta_4(\beta^2)} \right), \]  
where
\[ \theta_1(\nu) = 2p^{1/4}\sin(\pi\nu) \prod_{n=1}^{\infty} \left( 1 - p^{2n} \right) \left( 1 - e^{2\pi i\nu p^{2n}} \right) \left( 1 - e^{-2\pi i\nu p^{2n}} \right), \]
\[ \theta_4(\nu) = \prod_{n=1}^{\infty} \left( 1 - p^{2n} \right) \left( 1 - e^{2\pi i\nu p^{2n-1}} \right) \left( 1 - e^{-2\pi i\nu p^{2n-1}} \right) \]
and the prime in (6.2) denotes a derivative. The scaling limit of (6.1) is achieved by sending
\[ N \to \infty, \quad \varepsilon \to 0, \quad p \to 0 \]  
with the combinations
\[ R = Ne, \quad M = \frac{4}{\varepsilon} p^{(1+\xi)/4} \]
kept fixed. According to Refs. [24,22] in this limit the energy spectrum of (6.1) is described by the QFT (1.1), the parameter \( M \) coinciding with the sine-Gordon soliton mass.

In fact, the QFT (1.1) itself controls only the leading \( p \to 0 \) singularities in the spectrum of (6.1). Using exact XYZ ground state energy [23] one can easily extract subleading singular terms in this quantity. Being expressed through the scaling parameters \( M \) and \( R \), the singular at \( p \to 0 \) part of the bulk ground state energy reads
\[ (E_{XYZ})_{\text{sing}} = -\frac{RM^2}{4} \tan(\pi \xi/2) \left\{ 1 + \left( \frac{Me}{4} \right)^2 + O(\varepsilon^4) \right\} \]  
\( (RM \gg 1) \).  
Whereas the leading term here is the exact sine-Gordon vacuum energy, the higher order in \( \varepsilon \) terms must be attributed to the irrelevant operators which make the exact XYZ Hamiltonian (6.1) differ from the Hamiltonian \( H_{SG} \) of the sine-Gordon QFT (1.1). As follows from the analysis in [25], for \( \beta^2 < 2/3 \) the leading in \( \varepsilon \) correction comes from the terms
\[ H_{XYZ} = \text{const.} + H_{SG} - \frac{\varepsilon^2}{16} \int_0^R dx \left( \lambda_+ (\partial \varphi)^2 (\bar{\partial} \varphi)^2 + \lambda_- (\partial \varphi)^4 + (\bar{\partial} \varphi)^4 \right) + \ldots, \]
where \( \lambda_+ \) and \( \lambda_- \) are numerical coefficients whose exact values are found in [25] and the dots stand for the irrelevant operators of higher dimensions. The corrections in (6.5) can be expressed through the expectation values of the correction terms in (6.6) over
the sine-Gordon vacuum. Obviously, it is the VEV of the operator $(\partial \varphi)^2(\bar{\partial} \varphi)^2$ which is responsible for the $e^2$ term in (6.5) (the VEV of $(\partial \varphi)^4$ and $(\bar{\partial} \varphi)^4$ vanish), i.e.

$$\lambda_+((\partial \varphi)^2(\bar{\partial} \varphi)^2)_{SG} = \frac{M^4}{4} \tan(\pi \xi/2).$$

(6.7)

Using the result of Ref. [25]

$$\lambda_+ = -\frac{\cot(\pi \xi/2)}{2\pi},$$

(6.8)

one arrives precisely at Eq. (1.6), which therefore agrees with exact results of the lattice theory.

7. Application: two-point correlation function in scaling Lie–Yang model

In Ref. [3] a two-point correlation function in the so-called scaling Lee–Yang model (SLYM) was studied. In particular, a combination of the operator product expansions and conformal perturbation theory was used there to develop a short-distance expansion similar to (2.23). In this section we will use our result (1.8) to extend this expansion further thus obtaining a more accurate estimate for the two-point correlation function at all distances.

The SLYM is one of the simplest of the perturbed CFT (1.7), namely

$$A_{SLYM} = M_{2/5} + i\hbar \int d^2 x \Phi(x),$$

(7.1)

where

$$\Phi(x) = \Phi_{1,3}(x), \quad \Delta_\Phi = -\frac{1}{5}.$$

As is known (see, e.g., Ref. [15]) the QFT (7.1) is massive; it has one sort of massive particles whose mass $m$ is related to the parameter $\hbar$ in (7.1) as

$$h = \frac{2^{1/5}5^{3/4}}{16\pi^{6/5}} \frac{\Gamma(2/3)\Gamma(5/6)}{\Gamma(3/5)\Gamma(4/5)} m^{12/5} = 0.0970485 \ldots m^{12/5}.\quad (7.2)$$

We will use the notations

$$\Theta(x) = T_{\mu}^{\nu}(x)/4 = i\hbar \pi(1 - \Delta_\Theta) \Phi(x)$$

(7.3)

for the trace of the energy–momentum tensor associated with (7.1).

Consider the two-point correlation function

$$G(r) = \langle \Theta(x) \Theta(0) \rangle, \quad r = |x|.$$ 

(7.4)

According to Ref. [3] this correlation function admits the following short-distance expansion:
\[ G(r) = -\hbar^2 \pi^2 (1 - \Delta_\phi)^2 C_{\phi\phi}(r) + i\hbar \pi (1 - \Delta_\phi) C_{\phi\phi}(r) \langle \Theta \rangle \]
\[ -\hbar^2 \pi^2 (1 - \Delta_\phi)^2 C_{\phi\phi}^{\tau \tau}(r) \langle \tau \tau \rangle + O(r^{4/5}) \],

(7.5)

where the notation
\[ \tau \tau = L_2 \bar{L}_2 \]

(7.6)
is used. The coefficient functions \( C \) in (7.5) admit power series expansions in \( \hbar \), the first few terms being known explicitly \[3\] \[5\].

\[ C_{\phi\phi}^I(r) = r^{4/5} \left\{ 1 + \frac{5^{1/4}}{1960} \frac{\Gamma^4(1/5) \Gamma(3/5)}{\Gamma^3(4/5)} \hbar r^{12/5} + O(r^{24/5}) \right\}, \]

(7.7)

With the known exact VEV of the field \( \Theta \),
\[ \langle \Theta \rangle = -\frac{\pi}{4\sqrt{3}} m^2, \]

(7.8)

Eqs. (7.5), (7.7) effectively gives the short-distance expansion of the correlation function (7.4) up to the terms \( \sim r^{16/5} \) \[3\]. Now, using (7.8) we can derive the VEV
\[ \langle \tau \tau \rangle = -\frac{\pi^2}{48} m^4. \]

(7.9)

This additional piece of data allows one to compute explicitly the term \( \sim r^{24/5} \) in (7.5). The next term \( \sim r^{26/5} \), which would come from the \( \hbar^2 \) term in \( C_{\phi\phi}^\tau \), is still not available in an analytic form.

The correlation function (7.4) admits also the large-distance expansion in terms of exact form factors \[3\]. Two leading terms, corresponding to zero- and one-particle contributions, are known in analytic form,
\[ G(r) = \frac{\pi^2}{48} m^4 \left\{ 1 - \frac{27}{10\pi^2} Z K_0(mr) + \ldots \right\}, \]

(7.10)

where
\[ Z = \frac{10\sqrt{3}\pi}{27} \exp \left\{ - \int_0^{2\pi/3} \frac{dt}{\pi} \frac{t}{\sin(t)} \right\} = 0.8155740 \ldots, \]

Notice the analytic expression for the first subleading term in the expansion of \( C_{\phi\phi}^\tau \), which was given numerically in Ref. \[3\].
Table 1
Comparison of short- and long-distance expansions for the two-point correlation function (7.4). The first column gives the results of long-distance expansion which includes contributions of up to four-particle states (the four-particle contribution which we include here represents the improvement over the data in [31]). The data in the second and the third columns correspond to the short-distance expansion (7.5) without the $T\bar{T}$ term and with this term, respectively.

<table>
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<th>Long-distance expansion</th>
<th>Short-distance expansion</th>
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and $K_0(t)$ is the MacDonald function. Further terms in this expansion of $G(r)$ can be obtained by numerical integration of its spectral representation including the contributions of two or more particles in the intermediate state [3]. The expansion is known to converge very fast. With the inclusion of up to four-particle contributions this expansion gives a precision better then $10^{-2}\%$ for $mr \geq 10^{-2}$. The short-distance expansion (7.5) (with (7.7), (7.8) and (7.9)) is compared with this data in Table 1. The combined data from these two expansions apparently have relative precision $10^{-5}\%$ or better for all values of $r$. 
Finally, let us note that since exact form factors of the sine-Gordon model are known [26,27], similar numerical analysis can be performed for the general sine-Gordon correlation function (2.19).

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References