On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories

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We remark about an amusing universality in the thermodynamic Bethe ansatz equations for ADE-related diagonal scattering theories.

Recently there has been some amount of activity around the thermodynamic Bethe ansatz (TBA) approach to integrable relativistic field theory models (RFT) [1–5]. Originally this approach, which takes advantage of simplified factorized scattering in integrable theories to treat finite temperature effects, goes back to Yang and Yang [6]. Essentially the TBA is a relativistic version of the Yang and Yang technique. It starts with the factorized scattering of "physical" excitations in the effective massive RFT, which describes the scaling region near a second order phase transition point. Relativistic invariance then permits one to relate the finite temperature free energy to the Casimir effect and ultimately (in the high temperature region) to the parameters of the UV conformal field theory (CFT) [7]. That was the main purpose of the studies in refs. [1–5]. Along this line many reflectionless relativistic scattering theories were identified as the on-mass-shell data of different CFT's, perturbed by integrable relevant operators (the notion of integrable perturbation first appeared in ref. [8]).

In refs. [2,5] a remarkable set of diagonal scattering theories was classified and studied by means of the TBA, supporting their identification with particular integrable perturbations. All these scattering theories are related to simply-laced affine Lie algebras $\mathfrak{G}$ of the ADE series. Except for some special reduction in the series A_{2n}, they are the following:

The A_n series [9] is identified with the CFT's of $\mathbb{Z}_{n+1}$ parafermions [10] (these CFT's coincide with first representatives in the unitary series, associated with the $W(A_n)$ algebras [11]) with central charge $c(A_n)=2n/(n+3)$, perturbed by a primary field of dimension $\Delta(A_n)=2/(n+3)$ [12–16].

The D_n series S-matrices are that of the sine-Gordon scattering theory at the reflectionless points $\beta^2=8\pi/n$ [17]. Therefore the central charge here is independent of n: $c(D_n)=1$ and the perturbation is of dimension $\Delta(D_n)=1/n$.

The scattering theory related to E_6 [18,19] was supported to be a perturbation of the minimal model $\mathcal{M}(\frac{6}{7})$ (tricritical three-state Potts model) with $c(E_6)=\frac{6}{7}$, perturbed by the operator $\Phi_{(1,2)}$ of dimension $\Delta(E_6)=\frac{1}{7}$.

The E_7-related S-matrix [18,20,21] corresponds to the perturbation of $\mathcal{M}(\frac{4}{5})$ (tricritical Ising model) with $c(E_7)=\frac{7}{10}$ by a primary field $\Phi_{(1,2)}$ of dimension $\Delta(E_7)=\frac{1}{10}$.

The famous E_8 reflectionless scattering theory [22], which corresponds to the magnetic perturbation (dimension $\Delta(E_8)=\frac{1}{16}$) of the critical Ising model ($\mathcal{M}(\frac{3}{4}), c(E_8)=\frac{1}{2}$).

the number of particles $N(\mathfrak{G})$ in the $\mathfrak{G}$-related scattering theory coincides with the rank of $\mathfrak{G}$, their mass spectrum being proportional to the positive eigenvector of the corresponding Cartan matrix. The diagonal scattering amplitudes $S_{ab}(\theta)=S_{ba}(\theta)$, where the indices $a, b=1, ..., N(\mathfrak{G})$ run over the species of particles in the spectrum, are meromorphic functions of the rapidity difference θ and exhibit in the strip $0<\mathrm{Im}\,\theta<\pi$ a number of simple and multiple poles, equally spaced there by $\Delta\theta=2\pi i/h(\mathfrak{S})$. Here $h(\mathfrak{S})$ are the Coxeter numbers of the algebras $\mathfrak{S}$, listed below

$$\begin{aligned} &A_n \quad h=n+1\,, \\ &D_n \quad h=2n-2\,, \\ &E_6 \quad h=12\,, \\ &E_7 \quad h=18\,, \\ &E_8 \quad h=30\,. \end{aligned} \tag{1}$$

The TBA equations, as they come from the thermodynamic analyses of Bethe states, have the following form:

$$-\nu_a+\varepsilon_a+\frac{1}{2\pi}\sum_b \varphi_{ab}*\log[1+\exp(-\varepsilon_b)]=0\,, \quad a=1,\dots,N(\mathfrak{S})\,. \tag{2}$$

Here

$$\nu_a(\beta)=Rm_a\cosh\beta \tag{3}$$

are the energies (at rapidity β) of particles with the $\mathfrak{S}$-related mass spectrum m_a, and $\varepsilon_a(\beta)$ are β-dependent pseudoenergies, corresponding to each species of particles in the spectrum. In eq. (2) $\varphi_{ab}(\beta)$ denotes the symmetric matrix kernel, which encodes the scattering data

$$\varphi_{ab}(\beta)=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\beta}\log S_{ab}(\beta)\,, \tag{4}$$

and the star $*$ implies the rapidity convolution

$$\varphi*L=\int_{-\infty}^{\infty}\varphi(\beta-\beta')L(\beta')\,\mathrm{d}\beta'\,. \tag{5}$$

The pseudoenergies $\varepsilon_a(\beta)$ determine the Casimir energy $E(R)$ of the field theory on a circle of length R

$$E(R)=-\frac{1}{2\pi}\sum_a m_a\int_{-\infty}^{\infty}\cosh\beta\log\{1+\exp[-\varepsilon_a(\beta)]\}\,. \tag{6}$$

The construction implies much universality (as is usual for constructions subject to ADE classification) and is very suggestive for a more hidden perfection. The purpose of this paper is to make two remarks, which seem to add somewhat in this direction. First, it is amusing to observe that for all the ADE scattering theories discussed above the TBA equations can be transformed into the following universal form

$$-\nu_a+\epsilon_a+\frac{1}{\pi}\sum_b l_{ab}\varphi_h*\{\nu_b-\log[1+\exp(\varepsilon_b)]\}=0\,, \tag{7}$$

where l_{ab} is the incidence matrix of the $\mathfrak{S}$-related Dynkin diagram. The universal kernel $\varphi_n(\beta)$ depends only on the Coxeter number $h(\mathfrak{S})$

$$\varphi_h(\beta)=\frac{h}{2\cosh\frac{1}{2}h\beta}\,. \tag{8}$$

The equivalence between eqs. (2) and (7) is based on the following matrix relation:

$$\left(\delta_{ab}-\frac{1}{2\pi}\varphi_{ab}(k)\right)^{-1}=\delta_{ab}-\frac{1}{2\cosh(k/h)}l_{ab}\,, \tag{9}$$

which holds for the Fourier transforms

$$\varphi_{ab}(k)=\int_{-\infty}^{\infty}\varphi_{ab}(\beta)\exp(\mathrm{i}k\beta)\,\mathrm{d}\beta \tag{10}$$

of the kernel entries (4) and generalizes a similar relation for the total pole multiplicities in S_{ab}, quoted in refs. [2,5]. The relation

$$\sum_b l_{ab}m_b=2m_a\cos(\pi/h) \tag{11}$$

is also important in the derivation of eq. (7). Relation (9) seems to add some insight into the pole structure in the ADE scattering amplitudes.

A second remark is that relation (11) implies that the solution to eqs. (7) is also a particular solution of the following system of functional relations:

$$Y_a(\beta+\mathrm{i}\pi/h)Y_a(\beta-\mathrm{i}\pi/h)=\prod_b[1+Y_b(\beta)]^{l_{ab}}\,, \tag{12}$$

where $Y_a(\beta)=\exp[\varepsilon_a(\beta)]$. Note that eqs. (12) are completely independent on the form of the energies $\nu_a(\beta)$ (except for the relation $\nu_a(\beta+\mathrm{i}\pi/h)+\nu_a(\beta-\mathrm{i}\pi/h)=\sum_b l_{ab}\nu_b(\beta)$, which was used in the derivation of eq. (12)). Further, it is a matter of successive substitutions to convince oneself that eqs. (12) imply the following periodicity:

$$Y_a\left(\beta + i\pi \frac{h+2}{h}\right) = Y_a(\beta) \tag{13}$$

for the D_n and E series and

$$Y_a\left(\beta + i\pi \frac{h+2}{h}\right) = Y_{n-a+1}(\beta) \tag{14}$$

for the A_n series. This was verified for the A_n and D_n versions up to $n=8$ and also for E-related systems. Note that symmetry arguments in the A_n case require $Y_a(\beta) = Y_{n-a+1}(\beta)$, so that in fact eq. (13) is also satisfied.

The periodicity (13) has important consequences. First, it is easy to verify that all the function $Y_a(\beta)$ are entire functions of β, provided the $\nu_a(\beta)$ have the form (3). Therefore they admit the following Laurent expansions:

$$Y_a(\beta) = \sum_{n=-\infty}^{\infty} Y_a^{(n)} t^n , \tag{15}$$

with $t = \exp\{[2h/(h+2)]\beta\}$, which is convergent in the whole complex t-plane except for the points $t=0$ and $t=\infty$. In particular, for the solution to eq. (7) the symmetry under $\beta \to -\beta$ requires $Y_a^{(n)} = Y_a^{(-n)}$. In the t-plane the functional equations (12) acquire the form

$$Y_a(\Omega t) Y_a(\Omega^{-1} t) = \prod_b [1 + Y_b(t)]^{l_{ab}} , \tag{16}$$

where $\Omega = \exp[2i\pi/(h+2)]$. Naturally, eq. (16) admits a set of solutions which are entire functions of t. The "kink" solution (in the terminology of ref. [1]), corresponding to the energies $\nu_a(\beta) = Rm_a \exp(\beta)$ instead of (3), is obviously a representative of this set (this particular solution can be fixed by the special behavior near the essential singularity at $t=\infty$). Setting $t=0$ in eq. (16), one recovers the algebraic equation for the quantities $z_a = \exp[\varepsilon_a(0)]$, which are important in central charge calculations [1-5].

The second consequence of the periodicity (13) concerns the behavior of the solutions to eq. (2) in the high temperature limit $R \to 0$. In this limit all the functions $\log\{1 + \exp[-\varepsilon_a(\beta)]\}$ acquire the form of a plateau of approximately constant height $\log(1 + 1/z_a)$ in the central region $-\log(1/mR) \ll \beta \ll \log(1/mR)$ (here $m \sim m_a$ is the mass scale of the theory) and tend to zero very fast outside this region. As $R \to 0$ the plateau becomes wide and corrections to the constant height are of the form $\cosh([2hn/(h+2)]\beta)$ with integer $n = 1, 2, \ldots$. This implies that in the central region the integral equations (2) or (7) are in some sense local in the rapidity space and for R large the two edges of the plateau influence each other only via "waves" of wavelength predicted by the periodicity (13). Therefore the reduced Casimir energy $f(R) = RE(R)/2\pi$ expands in a regular series in $R^{4h/(h+2)}$ (except for a single bulk term, which is proportional to R^2) [1,2,5]. This structure is in complete agreement with the perturbation dimensions in the field theories, associated to the ADE diagonal scattering theories (note that in all the corresponding perturbation theories only even powers of the coupling constant contribute to the Casimir energy).

To conclude, the symmetries of the ADE-related TBA equations seem rather suggestive and encourage one to expect more exact results. All studies of refs. [1-5] were concerned with the reflectionless factorized scattering theories. Application of the same approach to factorized scattering with nonzero reflection amplitudes requires a higher-level Bethe ansatz technique [23]. ADE equations of the same form (7) again show up along these analyses, but with different energies ν_a. This is presumably a case for the kink scattering theories, corresponding to integrable perturbations of minimal CFT models $\mathcal{M}(p/(p+1))$, $p>3$, by means of the integrable operators $\Phi_{(1,3)}$ (TBA equations of A_{p-2}-related type), and for the sine-Gordon soliton scattering in the special points $\beta^2/8\pi = n/(n+1)$ (D_{n+1} type equations). The explicit form of these equations along with their analyses will be published elsewhere.

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